THE CONTINUOUS GIRDER.

HOWE.

THE THEORY

OF

THE CONTINUOUS GIRDER:

Its Application to Girders with and without Variable Cross-sections,

BY

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PREFACE.

Continuous bridges—with the exception of draw bridges—are not considered economical and are not designed by American engineers. This probably accounts for the brief treatment the theory of the continuous girder receives in text books and engineering literature.

With one or two exceptions, all American treatises consider the moment of inertia as constant and deduce *two* equations for the moment over any support, one to be applied when the loads are on the left of the support and the other when the loads are on the right. These two equations combined would, of course, give the moment over any support for any load, but only when the moment of inertia could be assumed as constant, as in girders with parallel flanges, where it might be undue refinement to consider the cross-section as variable.

American engineers have, until recently, assumed the moment of inertia as constant even when the flanges of the girder were considerably inclined, as in the Sabula draw bridge, relying upon the *factor of ignorance* to cover all discrepancies which might arise from the assumption. A few years ago the modulous of elasticity was quite variable in large bridges—though all engineers considered it as constant in computations—but now, since experience shows that iron and steel can be manufactured with, practically, a constant modulus of elasticity, it may safely be considered as constant, without leading to any appreciable error.

Although no material may be saved by considering the

moment of inertia as variable yet, if it is so considered, the material will be placed where it will do the best service.

Girders with inclined chords, computed and designed as if their cross-sections were constant, have more material than is necessary in some of the compression pieces, and not enough in some of the tension members, as is clearly shown by the results on page 77 of the text.

The object of the following pages is to present to computers, engineers and students a complete mathematical treatment of the theory of the continuous girder, and show how it can be applied to any girder—especially to fixed girders and girders of two and three spans—under any conditions.

With the single assumption that the modulous of elasticity is constant a general equation has been deduced for the moment over any support of any girder under any conditions of loading, of any length of spans, of variable or constant cross-section and for supports at any level.—By difference of level of the supports is meant any change of level which may take place when the girder is in position. It is evident that such a change alone would affect the moments.—From this equation special equations are readily deduced for any particular case. To illustrate the simplicity of the transformations necessary for any special case and also for the convenience of engineers, equations have been given for all the special cases usually discussed in text books, and also equations for these cases when the moment of inertia is considered as variable.

The usual general equations for reactions, deflections and intermediate moments and their transformed equations for special cases are also presented.

Several examples are solved to illustrate the application of the formulas and to show that the processes are almost mechanical when the formulas are thoroughly understood. From those examples which are solved considering the moment of inertia as constant and then variable, a good idea of the manner in which the moment of inertia affects the

results can be obtained. In the case of the Sabula draw there is a difference of about twenty per cent.

In all pin connected bridges the loads are considered to be concentrated at the apices or panel points. In computing the moments for such girders Table I. will be found to be very convenient, as in it are found expressions for $2k-3k^2+k^3$ and $k-k^3$ for all values of $k=\frac{a}{l}$ from 0.001 to 0.999 inclusive.

The works enumerated under References, page 107, have been consulted, and some of their parts used without any material change, for which credit is given in foot notes.

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M. A. H.

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TABLE I.

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SUMMARY.

NOMENCLATURE.

r=The number of the support just at the *left* of the r^{th} span.

l_r=The horizontal length of the span r.

P_r=Any concentrated load in the r^{th} span.

w_r=Any uniform load, per lineal foot, in the r^{th} span.

a_r=The distance from the left support r to any concentrated load P_r. $a_r = k_r\, l_r$.

a_r'=The distance from the support r to the point where the uniform load ends in the r^{th} span.

a_r''=The distance from the left support r to the point where the uniform load begins in the r^{th} span.

$k_r = \dfrac{a_r}{l_r}$, or, $a_r = k_r\, l_r$.

x_r=The distance from the left support r to any point in the r^{th} span.

M_r=The bending moment over the support r.

M_m=The bending moment over any support m.

$\overset{r}{M}_x$=The bending moment at any section, x_r from the support r.

M_c=The bending moment at the center of any span of a girder.

S_r=The shear just at the *right* of the support r.

S_r'=The shear just at the *left* of the support r.

R_r=The reaction at the support r, and equals $S_r + S_r'$.

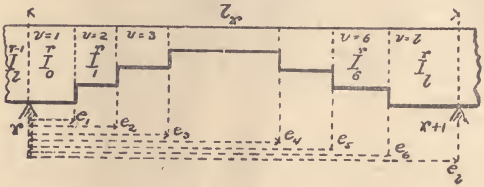
h_r=The distance the support r is below some specified horizontal line of reference.

Σ=Sign of summation.

t_r=Tangent of the angle that the elastic line makes with the horizontal over the support r.

y_r=The deflection of the girder at any section x_r; y_r is measured from the horizonal line of reference.

s=The number of spans.



e_1=The distance from the left support r to the point where the moment of inertia of the section of the girder changes for the first time in the span r.

e_2=The distance to the point where it changes the second time.

e_v=The distance to any point where the moment of inertia changes, always measured from the left support r and in the span r.

$\overset{r}{I}_l$=The moment of inertia between the last value of e_v and the end of the r^{th} span. See Fig.

$\overset{r}{I}_0$=The moment of inertia between the support r and the point e_1.

$\overset{r}{I}_1$=The moment of inertia between e_1 and e_2 in the r^{th} span.

$\overset{r}{I}_x$=The moment of inertia at any section in the r^{th} span.

E=The modulus of elasticity.

GENERAL RELATIONS.

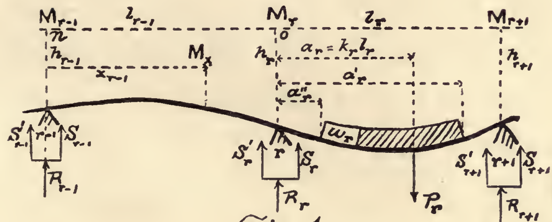


Fig. 1.

* In the r^{th} span of a continuous girder, whose length is l_r, Fig. 1, take a point o vertically above the r^{th} support, as the origin of co-ordinates, and the horizontal through o as the axis of abscissas. Suppose any section of the girder at a distance x_r from the left support r, and between this section and the support r a weight P_r distant from r, $a_r = k_r\ l_r$.

Now, if the girder is continuous over any number of supports there will be a bending moment over each of them (if the ends are not fixed the bending moment over the first and last support will equal zero): M_{r-1} over $r-1$, M_r over r, M_{r+1} over $r+1$, etc., also, there will be a shear S_r just at the right and S_r' just at the left of r, S_{r+1}, just at the right and S'_{r+1} just at the left of $r+1$, etc. If there is equilibrum the following conditions must be fulfilled :

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* See "Continuirlichen Und Einfachen Träger," page 4, by Prof. Weyrauch.
"Theory and Calculation of Continuous Bridges," page 52, by Prof. Merriman.
"Strains in Framed Structures," page 244, by Prof. DuBois.

I. The algebraic sum of all the horizontal forces must be zero.

II. The algebraic sum of all the vertical forces must be zero.

III. The algebraic sum of the moments of all the forces must be zero.

From the third condition we have for any section in the r^{th} span,

$$\overset{r}{M_x} = M_r + S_r\, x_r - P_r\,(x_r - a_r) \quad \ldots \quad x_r \gtreqless a_r \quad \ldots \quad (1)$$

If in (1) we make $x_r = l_r$, $\overset{r}{M_x} = M_{r+1}$, and it becomes

$M_{r+1} = M_r + S_r\, l_r - P_r\,(l_r - a_r)$, or, since $a_r = k_r\, l_r$, we have

$$M_{r+1} = M_r + S_r\, l_r - P_r\, l_r\,(1 - k_r) \quad \ldots \quad (2)$$

From (2),

$$S_r = \frac{M_{r+1} - M_r}{l_r} + P_r\,(1 - k_r) \quad \ldots \quad (3)$$

And if there is no load in the span r this becomes

$$S_r = \frac{M_{r+1} - M_r}{l_r} \quad \ldots \quad (4)$$

$S'_{r+1} = P_r - S_r$, therefore from (3),

$$S'_{r+1} = \frac{M_r - M_{r+1}}{l_r} + P_r\, k_r, \text{ or better,}$$

$$S'_r = \frac{M_{r-1} - M_r}{l_{r-1}} + P_{r-1}\, k_{r-1} \quad \ldots \quad (5)$$

And if there is no load in the span $r-1$, (5) becomes

$$S'_r = \frac{M_{r-1} - M_r}{l_{r-1}} \quad \ldots \quad (6)$$

The reaction at any support equals the sum of the shears at that support, or

$$R_r = S_r + S'_r \quad \ldots \quad (7)$$

The above formulas were deduced under the supposition that there was but a single concentrated load $\boldsymbol{P}_r$ in the span $\boldsymbol{r}$. If there be more than one concentration, the formulas become:

$$(1) \quad \overset{r}{M_r} = M_r + S_r\, x_r - \Sigma\, P_r\, (x_r - a_r) \quad . \quad . \quad . \quad x_r \gtreqless a_r \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$(2) \quad M_{r+1} = M_r + S_r\, l_r - \Sigma\, P_r\, l_r\, (1 - k_r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$(3) \quad S_r = \frac{M_{r+1} - M_r}{l_r} + \Sigma\, P_r\, (1 - k_r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$(5) \quad S_r' = \frac{M_{r-1} - M_r}{l_{r-1}} + \Sigma\, P_{r-1}\, k_{r-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Any partial uniform load in span r.—If $\boldsymbol{w}_r$ represents the uniform load per lineal foot in the span $\boldsymbol{r}$, we can write

$$\Sigma\, P_r = \int_{a_r = a_r''}^{a_r = a_r'} w_r\, da_r \text{ or, since } a_r = k_r\, l_r$$

$$\Sigma\, P_r = \int_{a_r'' = k_r\, l_r}^{a_r' = k_r\, l_r} w_r\, l_r\, d\, k_r = (w_r\, l_r\, k_r) \Bigg|_{k_r = \frac{a_r''}{l_r}}^{k_r = \frac{a_r'}{l_r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The last expression in (12) indicates the difference of the values of the parenthesis when $\boldsymbol{k}_r$ equals $\frac{\boldsymbol{a}_r'}{\boldsymbol{l}_r}$ and $\frac{\boldsymbol{a}_r''}{\boldsymbol{l}_r}$ respectively.

Substituting (12) in the above equations, we have,

From (8),

$$\overset{r}{M_r} = M_r + S_r\, x_r - \int_{a_r''}^{a_r'} w_r\, l_r\, d\, k_r\, (x_r - k_r\, l_r) \quad . \quad . \quad . \quad x_r \gtreqless a_r' \quad . \quad . \quad . \quad . \quad (13)$$

From (9),

$$M_{r+1}=M_r+S_r\ l_r-w_r\ l_r^2\int_{a''_r}^{a'_r} d\ k_r\ (1-k_r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Or

$$M_{r+1}=M_r+S_r\ l_r-w_r\ l_r^2\ \left(k_r-\frac{1}{2}\ k_r^2\right)\Bigg|_{k_r=\frac{a''_r}{l_r}}^{k_r=\frac{a'_r}{l_r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

From (10),

$$S_r=\frac{M_{r+1}-M_r}{l_r}+w_r\ l_r\ \left(k_r-\frac{1}{2}\ k_r^2\right)\Bigg|_{k_r=\frac{a''_r}{l_r}}^{k_r=\frac{a'_r}{l_r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

From (11),

$$S'_r=\frac{M_{r-1}-M_r}{l_{r-1}}+\frac{1}{2}\ w_{r-1}\ l_{r-1}\ k_{r-1}^2\Bigg|_{k_{r-1}=\frac{a''_{r-1}}{l_{r-1}}}^{k_{r-1}=\frac{a'_{r-1}}{l_{r-1}}} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Uniform load over entire span r.—If the entire span is uniformly loaded, we have $a'_r=l_r$, $a''_r=o$ and $a_r=\frac{1}{2}\ l_r$.

From (13), (In this case $a'_r=x_r$, $a''_r=o$ and $a_r=\frac{1}{2}\ x_r$ since the load considered cannot extend beyond x_r).

$$M_r^r=M_r+S_r\ x_r-\frac{1}{2}\ w_r\ x_r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

From (15),

$$M_{r+1}=M_r+S_r\ l_r-\frac{1}{2}\ w_r\ l_r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

From (16),

$$S_r = \frac{M_{r+1} - M_r}{l_r} + \frac{1}{2} w_r \, l_r \quad \ldots \ldots \ldots \ldots \quad (20)$$

From (17),

$$S'_r = \frac{M_{r-1} - M_r}{l_{r-1}} + \frac{1}{2} w_{r-1} \, l_{r-1} \quad \ldots \ldots \ldots \ldots \quad (21)$$

These twenty-one general equations apply to all spans and conditions of loading, but before they can be solved, it will be necessary to find expressions for $\boldsymbol{M}$ or $\boldsymbol{S}$ in terms of the loads, lengths of the spans, etc.

This can be done by introducing the equation of the *elastic line* and the theorem of *three moments*. The following expression is obtained for the bending moment over any support $\boldsymbol{m}$; $\boldsymbol{E}$ *alone, being constant.*

$$*M_m = \frac{-c_m \, \beta_2 \, \beta_3 \ldots \beta_{m-1}}{(d_{s+1} \; \beta''_1) \, \beta''_2 \ldots \beta''_{m-1}} \sum_{r=s}^{r=m} \left\{ (A_r + Y_r + X'_r + X''_{r-1}) d_{s-r+2} + B_r d_{s-r+1} \right\}$$

$$+ \frac{-d_{s-m+2} \, \beta''_{s-1} \; \beta''_{s-2} \ldots \beta''_m}{(c_{s+1} \; \beta_s) \; \beta_{s-1} \; \beta_{s-2} \ldots \beta_m} \sum_{r=m-1}^{r=1} \left\{ (A_r + Y_r + X'_r + X''_{r-1}) \, c_r + B_r \, c_{r+1} \right\} \quad (A)$$

$$\beta''_{s-1} \underset{>}{=} \beta''_m . \quad \beta_{s-1} \underset{>}{=} \beta_m . \quad \beta_2 \underset{<}{=} \beta_{m-1} . \quad \beta''_2 \underset{<}{=} \beta''_{m-1} .$$

In which the respective expressions have the following values:

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* A complete demonstration of this equation is given in the Appendix and should be thoroughly understood before any attempt is made to apply equation **A** practically.

A_r

Concentrated loads—

$$A_r = -\Sigma\, P_r\, l_r^2\, (2\, k_r - 3\, k_r^2 + k_r^3)\, \theta_{r-1} \quad \text{. . . . See (57) . . . .} \quad (i)$$

Partial uniform load—

From (12),

$$\Sigma\, P_r = w_r\, l_r\, (k_r)\Big|_{k_r = \frac{a''_r}{l_r}}^{k_r = \frac{a'_r}{l_r}} \quad \text{therefore } (i) \text{ becomes}$$

$$A_r = -w_r\, l_r^3 \left\{ k_r^2 - k_r^3 + \frac{k_r^4}{4} \right\}_{a''_r = k_r\, l_r}^{a'_r = k_r\, l_r} \theta_{r-1} \quad \text{. . . . .} \quad (74)$$

Uniform load over all—

$$A_r = -\frac{1}{4}\, w_r\, l_r^3\, \theta_{r-1} \quad \text{. . . . .} \quad (75)$$

B_r

Concentrated loads—

$$B_r = -\Sigma\, P_r\, l_r^2\, (k_r - k_r^3)\, \theta_{r+1} \quad \text{. . . . . . . See (58) . . . .} \quad (j)$$

Partial uniform loads—

$$B_r = -w_r\, l_r^3 \left\{ \frac{2\, k_r^2 - k_r^4}{4} \right\}_{a''_r = k_r\, l_r}^{a'_r = k_r\, l_r} \theta_{r+1} \quad \text{. . . . .} \quad (76)$$

Uniform load over all—

$$B_r = -\frac{1}{4}\, w_r\, l_r^3\, \theta_{r+1} \quad \text{. . . . .} \quad (77)$$

X'_r

Concentrated loads—

$$X'_r = -F'_r \; \Sigma \; P_r \; l_r \; (1-k_r) \; \theta_{r-1} + H_r \; \theta_{r-1} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \; (h)$$

Partial uniform load—

$$X'_r = -F'_r \; w_r \; l_r^2 \; \theta_{r-1} \left\{ k_r - \frac{1}{2} \; k_r^2 \right\}_{a''_r = k_r \, l_r}^{a'_r = k_r \, l_r} + H_r \; \theta_{r-1} \; . \; . \; (78)$$

Uniform load over all—

$$X'_r = -\frac{1}{2} \; F'_r \; w_r \; l_r^2 \; \theta_{r-1} + H_r \; \theta_{r-1} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \; (79)$$

X''_{r-1}

Concentrated loads—

$$X''_{r-1} = -2 \; F''_{r-1} \; \Sigma \; P_{r-1} \; l_{r-1} \; (1-k_{r-1}) \; \theta_r - H'_{r-1} \; \theta_r \ldots \text{See (59)} \ldots (h)$$

Partial uniform load—

$$X''_{r-1} = -2 \; F''_{r-1} \; \theta_r \; w_{r-1} \; l_{r-1}^2 \left\{ k_{r-1} - \frac{1}{2} k_{r-1}^2 \right\}_{a''_{r-1} = k_{r-1} \, l_{r-1}}^{a'_{r-1} = k_{r-1} \, l_{r-1}} - H'_{r-1} \; \theta_r \; . \; . \; (80)$$

Uniform load over all—

$$X''_{r-1} = - \; F''_{r-1} \; \theta_r \; w_{r-1} \; l_{r-1}^2 - H'_{r-1} \; \theta_r \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \; (81)$$

H_r

Concentrated loads—

$$H_r = \sum_{v = l_r}^{r=1} \underset{v}{\overset{r}{\triangle}} \left\{ \frac{\Sigma \; P_r \; (e_v - a_r)^3}{l_r} + \frac{3 \; (l_r - e_v)}{l_r} \Sigma \; P_r \; (e_v - a_r)^2 \right\} \; . \; . \; . \; (f)$$

Partial uniform load—

Substitute the following expressions in (f):

$$\Sigma\ P_r\ (e_v-a_r)^3=\int_{a''_r}^{a_r} w_r\ d\ a_r\ (e_v-a_r)^3\ .\ .\ .\ .\ a'_r\leqq e_v\ .\ .\ .\ .\ .\ (82)$$

$$\Sigma\ P_r\ (e_v-a_r)^2=\int_{a''_r}^{a'_r} w_r\ d\ a_r\ (e_v-a_r)^2\ .\ .\ .\ .\ a'_r\leqq e_v\ .\ .\ .\ .\ .\ (83)$$

Probably the easiest way to handle this case is to consider the uniform load as extending from the left support to the farther end of the load and integrate between the limits $a'_r\leqq e_v$ and $a''_r=o$; then consider the load as extending from the left support to the nearer end of the load and integrate between the limits $a'_r=a''_r\leqq e_v$ and o, and take the difference of the results.

Uniform load over all—

$$\Sigma\ P_r\ (e_v-a_r)^3=\int_{o}^{e_v} w_r\ d\ a_r\ (e_v-a_r)^3=\frac{1}{4}w_r\ e_v^4\ .\ .\ .\ .\ .\ .\ .\ (84)$$

$$\Sigma\ P_r\ (e_v-a_r)^2=\int_{o}^{e_v} w_r\ d\ a_r\ (e_v-a_r)^2=\frac{1}{3}w_r\ e_v^3\ .\ .\ .\ .\ .\ .\ .\ (85)$$

H'_r

Concentrated loads—

$$H'_r=\sum_{v=l_r}^{v=1} \triangle_v^r \left\{\frac{\Sigma\ P_r\ (e_v-a_r)^3}{l_r}-\frac{3\ e_v}{l_r}\ \Sigma\ P_r\ (e_v-a_r)^2\right\}\ .\ .\ .\ (g)$$

Partial uniform load—

$$\Sigma\, P_r\,(e_v-a_r)^3=\int_{a_r''}^{a_r'} w_r\ d\ a_r\ (e_v-a_r)^3\ \ .\ \ .\ \ .\ \ a_r'\leqq e_v\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ (82)$$

$$\Sigma\, P_r\,(e_v-a_r)^2=\int_{a_r''}^{a_r'} w_r\ d\ a_r\ (e_v-a_r)^2\ \ .\ \ .\ \ .\ \ a_r'\leqq e_v\ \ .\ \ .\ \ .\ \ (83)$$

Uniform load over all—

$$\Sigma\, P_r\,(e_v-a_r)^3=\int_{0}^{e_v} w_r\ d\ a_r\ (e_v-a_r)^3=\frac{1}{4}\,w_r\ e_v^4\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ (84)$$

$$\Sigma\, P_r\,(e_v-a_r)^2=\int_{0}^{e_v} w_r\ d\ a_r\ (e_v-a_r)^3=\frac{1}{3}\,w_r\ e_v^3\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ (85)$$

For all loads—

$$F_r=\sum_{v=l_r}^{v=1}\overset{r}{\underset{v}{\triangle}}\left\{e_v\left(3-\frac{3\,e_v}{l_r}+\frac{e_v^2}{l_r^2}\right)=\frac{l_r^3-(l_r-e_v)^3}{l_r^2}\right\}\text{ See (48) . . } (b)$$

$$F_r'=\sum_{v=l_r}^{v=1}\overset{r}{\underset{v}{\triangle}}\ e_v^2\left\{\frac{3}{l_r}-\frac{2\,e_v}{l_r^2}\right\}\ .\ .\ .\ .\ .\ \text{ See (49) . . . . . . } (c)$$

$$F_r''=\sum_{v=l_r}^{v=1}\overset{r}{\underset{v}{\triangle}}\ e_v^3\,\frac{1}{l_r^2}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \text{ See (50) . . . . } (d)$$

$$\overset{r}{\underset{v}{\triangle}}=\frac{\overset{r}{I_l}}{\underset{v-1}{\overset{r}{I}}}-\frac{\overset{r}{I_l}}{\underset{v}{\overset{r}{I}}}\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ (a)$$

$\beta_r = (l_r + F_r')\ \theta_{r+1}$ . . . . See (60) . . . . . (m)

$\beta_r' = (l_{r-1} + F_{r-1}'')\ \theta_r + (l_r + F_r)\ \theta_{r-1}$ . . . See (61) . . . . . (n)

$\beta_r'' = (l_r + F_r')\ \theta_{r-1}$ . . . . See (62) . . . . . (o)

$$Y_r = -\theta_r\ \theta_{r-1} \left\{ \frac{h_r - h_{r-1}}{l_{r-1}} + \frac{h_r - h_{r+1}}{l_r} \right\} \quad \ldots \text{ See (56)} \ldots \quad (l)$$

$\theta_r = 6\ E\ I_l^r$ . . . . See (51) . . . . . (e)

$c_1 = o. \qquad c_2 = 1.$

$$c_m = -2\ c_{m-1} \frac{\beta_{m-1}'}{\beta_{m-1}} - c_{m-2} \frac{\beta_{m-2}''}{\beta_{m-1}} \quad \ldots \text{ See (66)} \ldots \quad (p)$$

$d_1 = o. \qquad d_2 = 1.$

$$d_m = -2\ d_{m-1} \frac{\beta_{s-m+3}'}{\beta_{s-m+2}''} - d_{m-2} \frac{\beta_{s-m+3}}{\beta_{s-m+2}''} \quad \ldots \text{ See (68)} \ldots \quad (r)$$

By means of the above equations we can deduce the bending moment over any support of any continuous girder under the single assumption that the *modulus of elasticity*, ***E***, *shall be constant.*

E AND I CONSTANT.

If the moment of inertia as well as the modulus of elasticity is constant our equations will be much more simple.

By inspection, we see that (a) equals zero, and hence F_r, F_r' and F_r'' equal zero and also H_r and H_r'; hence X_r' and X_{r-1}'' equal zero. By reduction we obtain

$$M_m = \frac{-c_m}{d_{s+1}\ l_1} \sum_{r=s}^{r=m} \left\{ (A_r + Y_r)\ \ d_{s-r+2} + B_r\ \ d_{s-r+1} \right\}$$

$$+ \frac{-d_{s-m+2}}{c_{s+1}\ l_s} \sum_{r=m-1}^{r=1} \left\{ (A_r + Y_r)\ \ c_r + B_r\ \ c_{r+1} \right\} \quad \ldots \quad (A_1)$$

$c_1 = o. \quad c_2 = 1.$

$$c_m = -2\, c_{m-1} \frac{l_{m-2} + l_{m-1}}{l_{m-1}} - c_{m-2} \frac{l_{m-2}}{l_{m-1}} \quad \ldots \ldots \ldots \ldots (p_1)$$

$d_1 = o. \quad d_2 = 1.$

$$d_m = -2\, d_{m-1} \frac{l_{s-m+3} + l_{s-m+2}}{l_{s-m+2}} - d_{m-2} \frac{l_{s-m+3}}{l_{s-m+2}} \quad \ldots \ldots \ldots (r_1)$$

$$Y_r = -6\, E\, I \left\{ \frac{h_r - h_{r-1}}{l_{r-1}} + \frac{h_r - h_{r+1}}{l_r} \right\} \quad \ldots \ldots \ldots \ldots (l_1)$$

$\boldsymbol{A_r}$

Concentrated loads—

$$A_r = -\Sigma\, P_r\, l_r^2\, (2\, k_r - 3\, k_r^2 + k_r^3) \quad \ldots \ldots \ldots \ldots (i_1)$$

Partial uniform load—

$$A_r = -w_r\, l_r^3 \left\{ k_r^2 - k_r^3 + \frac{k_r^4}{4} \right\}_{a''_r = k_r\, l_r}^{a'_r = k_r\, l_r} \quad \ldots \ldots \ldots \ldots (86)$$

Uniform load over all—

$$A_r = -\frac{1}{4} w_r\, l_r^3 \quad \ldots \ldots \ldots \ldots \ldots (87)$$

$\boldsymbol{B_r}$

Concentrated loads—

$$B_r = -\Sigma\, P_r\, l_r^2\, (k_r - k_r^3) \quad \ldots \ldots \ldots \ldots \ldots (j_1)$$

Partial uniform load—

$$B_r = -w_r\, l_r^3 \left\{ \frac{2\, k_r^2 - k_r^4}{4} \right\}_{a''_r = k_r\, l_r}^{a'_r = k_r\, l_r} \quad \ldots \ldots \ldots \ldots (88)$$

Uniform load over all—

$$B_r = -\frac{1}{4} w_r\, l_r^3 \quad \ldots \ldots \ldots \ldots \ldots (89)$$

E AND I CONSTANT—Spans equal.

If the spans are equal, the equations of the last case reduce to,

$$M_m = \frac{-c_m}{d_{s+1}\, l} \sum_{r=s}^{r=m} \left\{ (A_r + Y_r)\, d_{s-r+2} + B_r\, d_{s-r+1} \right\}$$

$$+ \frac{-d_{s-m+2}}{c_{s+1}\, l} \sum_{r=m-1}^{r=1} \left\{ (A_r + Y_r)\ c_r + B_r\ c_{r+1} \right\} \quad . \quad . \quad . \quad (A_2)$$

$c_1 = o. \quad c_2 = 1.$

$$c_m = -4\, c_{m-1} - c_{m-2} \quad . \quad . \quad . \quad (p_2)$$

$d_1 = o. \quad d_2 = 1.$

$$d_m = -4\, d_{m-1} - d_{m-2} \quad . \quad . \quad . \quad (r_2)$$

A_r

Concentrated loads—

$$A_r = -\Sigma\, P_r\, l^2\, (2\, k_r - 3\, k_r^2 + k_r^3) \quad . \quad . \quad . \quad (i_2)$$

Partial uniform load—

$$A_r = -\, w_r\, l^3 \left\{ k_r^2 - k_r^3 + \frac{k_r^4}{4} \right\}_{a_r''=k_r\, l}^{a_r'=k_r\, l} \quad . \quad . \quad . \quad (90)$$

Uniform load over all—

$$A_r = -\frac{1}{4}\, w_r\, l^3 \quad . \quad . \quad . \quad (91)$$

B_r

Concentrated loads—

$$B_r = -\Sigma\, P_r\, l^2\, (k_r - k_r^3) \quad . \quad . \quad . \quad (j_2)$$

Partial uniform load—

$$B_r = -\, w_r\, l^3 \left\{ \frac{2\, k_r^2 - k_r^4}{4} \right\}_{a_r''=k_r\, l}^{a_r'=k_r\, l} \quad . \quad . \quad . \quad (92)$$

Uniform load over all—

$$B_r = -\frac{1}{4}\,w_r\,l^3 \quad . \quad . \quad . \quad (93)$$

$$Y_r = -6\,E\,I\left\{\frac{2\,h_r - h_{r-1} - h_{r+1}}{l}\right\} \quad . \quad . \quad . \quad (l_2)$$

In case the supports are at the same level, $Y_r = o$ in all equations.

From (p_2) and (r_2) we obtain the following values for c and d:

$c_1 = \pm\ \ 0 = d_1$	$c_7 = -\ \ 780 = d_7$
$c_2 = +\ \ 1 = d_2$	$c_8 = +\ \ 2911 = d_8$
$c_3 = -\ \ 4 = d_3$	$c_9 = -\ \ 10864 = d_9$
$c_4 = +\ \ 15 = d_4$	$c_{10} = +\ \ 40545 = d_{10}$
$c_5 = -\ \ 56 = d_5$	$c_{11} = -\ \ 151316 = d_{11}$
$c_6 = +\ 209 = d_6$	$c_{12} = +\ 564719 = d_{12}$

Explanation of Table I— In the co-efficients $(2\,k_r - 3\,k_r^2 + k_r^3)$ and $(k_r - k_r^3)$, $\boldsymbol{k_r}$ is a fraction and equals $\frac{a_r}{l_r}$, hence, it is immaterial about the actual values of a and l as long as the *ratio* is known.

The ratio $\boldsymbol{k_r}$ has been assumed to have all possible values from ***.001*** to ***.999*** inclusive, and the respective values of the above co efficients, carefully computed, are given in Table I. A few trials will prove the great utility of this table, and convince the computer that much time and labor can be saved by its use.

SHEAR.

The moments over the supports being determined, the shears can be found from the following formulas, which apply to *all cases.*

$$S_r = \frac{M_{r+1} - M_r}{l_r} + Q_r \quad . \quad . \quad . \text{ See (10)} \quad . \quad . \quad . \quad (B)$$

$$S'_r = \frac{M_{r-1} - M_r}{l_{r-1}} + Q'_{r-1} \quad \ldots\ldots\ldots \text{ See (11)} \ldots\ldots (C)$$

In which Q and Q' have the following values:

Concentrated loads—

$$Q_r = \Sigma\, P_r\, (1 - k_r) \quad \ldots\ldots\ldots\ldots (94)$$

$$Q'_r = \Sigma\, P_r\, k_r \quad \ldots\ldots\ldots\ldots (95)$$

Partial uniform load—

$$Q_r = w_r\, l_r \left\{\frac{2\,k_r - k_r^2}{2}\right\}_{a''_r = k_r\, l_r}^{a'_r = k_r\, l_r} \quad \ldots\ldots\ldots\ldots (96)$$

$$Q'_r = w_r\, l_r \left\{\frac{k_r^2}{2}\right\}_{a''_r = k_r\, l_r}^{a'_r = k_r\, l_r} \quad \ldots\ldots\ldots\ldots (97)$$

Uniform load over all—

$$Q_r = \frac{1}{2}\, w_r\, l_r \quad \ldots\ldots\ldots\ldots (98)$$

$$Q'_r = \frac{1}{2}\, w_r\, l_r \quad \ldots\ldots\ldots\ldots (99)$$

INTERMEDIATE BENDING MOMENTS.

General equations for *all possible cases.*

$$\overset{r}{M_x} = M_r + S_r\, x_r - L_r \quad \ldots\ldots\ldots \text{ See (8)} \ldots\ldots (D)$$

In which L has the following values:

Concentrated loads—

$$L_r = \Sigma\, P_r\, (x_r - a_r) \quad \ldots\ldots\ldots \quad a_r \leqq x_r \quad \ldots\ldots (100)$$

Partial uniform loads—

$$L_r = w_r \left\{a_r\left(x_r - \frac{a_r}{2}\right)\right\}_{a_r = a''_r}^{a_r = a'_r} \quad \ldots\ldots \quad a'_r \leqq x_r \quad \ldots\ldots (101)$$

$$L_r = w_r \left\{a_r\left(x_r - \frac{a_r}{2}\right)\right\}_{a'_r = a''_r = o}^{a_r = a'_r = x_r} = \frac{1}{2}\, w_r\, x_r^2 \quad \ldots\ldots (102)$$

DEFLECTION.

General formula. $\boldsymbol{E}$ alone constant.

$$y_r = h_r + t_r\ x_r + \frac{1}{6\ E\ \overset{r}{I_x}} \left\{ 3\ M_r\ x_r^2 + S_r\ x_r^3 - \Sigma\ P_r (x_r - a_r)^3 \right\}$$

$$+ \frac{1}{6\ E\ \overset{r}{I_x}} \sum_{v=x_r}^{v=1} \left(\frac{\overset{r}{I_x}}{\overset{r}{I_{v-1}}} - \frac{\overset{r}{I_x}}{\overset{r}{I_v}} \right) \left\{ 3\ M_r\, e_v\, (2\, x_r - e_v) + S_r\, e_v^2\ (3\, x_r - 2\, e_v) \right.$$

$$\left. - \Sigma\ P_r\ (e_v - a_r)^3 - 3\ (x_r - e_v)\ \Sigma\ P_r\ (e_v - a_r)^2 \right\}\ . \text{ See (39) . . } (E)$$

$$t_{r+1} = \frac{h_{r+1} - h_r}{l_r} + \frac{1}{6\ E\ \overset{r}{I_l}} \left\{ M_r\ l_r + 2\ M_{r+1}\ l_r + \Sigma\, P_r\, l_r^2\, (k_r - k_r^3) \right\}$$

$$+ \frac{1}{6\, E\, \overset{r}{I_l}\, l_r} \sum_{v=l_r}^{v=1} \left(\frac{\overset{r}{I_l}}{\overset{r}{I_{v-1}}} - \frac{\overset{r}{I_l}}{\overset{r}{I_v}} \right) \left\{ M_r\, e_v^2\ (3 - \frac{2\, e_v}{l_r}) + 2\, M_{r+1} \frac{e_v^3}{l_r} \right.$$

$$\left. + \frac{2\, e_v^3}{l_r}\ \Sigma\ P_r\ l_r\ (1 - k_r) + \Sigma\ P_r\ (e_v - a_r)^3 - 3\, e_v\ \Sigma\, P_r\ (e_v - a_r)^2 \right\} . . (45)$$

For uniform loads—

$$\Sigma\ P_r = \int_{a_r = a_r''}^{a_r = a_r'} w_r\ d\ a_r = \int_{a_r'' = k_r\ l_r}^{a_r' = k_r\ l_r} w_r\ l_r\ d\ k_r \quad . . . . (12)$$

Remember that in the terms containing $(x - a)$ and $(e - a)$, $\boldsymbol{x}$ and $\boldsymbol{e}$ must never be *less* than $\boldsymbol{a}$, or rather $\boldsymbol{a} \leq \boldsymbol{x}$ or $\boldsymbol{e}$.

E AND I CONSTANT.

If the moment of inertia is constant the preceding equations reduce to,

$$y_r = h_r + t_r\ x_r + \frac{1}{6\ E\ I} \left\{ 3\, M_r\ x_r^2 + S_r\ x_r^3 - \Sigma\, P_r\ (x_r - a_r)^3 \right\} \ . \ . \ (E_1)$$

$$t_{r+1} = \frac{h_{r+1} - h_r}{l_r} + \frac{1}{6\ E\ I} \left\{ M_r\ l_r + 2\ M_{r+1}\ l_r + \Sigma\ P_r\ l_r^2\ (k_r - k_r^3) \right\} . . . (45_a)$$

For uniform loads—

$$\Sigma\ P_r = \int_{a_r=a''_r}^{a_r=a'_r} w_r\ d\ a_r = \int_{a''_r=k_r\ l_r}^{a'_r=k_r\ l_r} w_r\ l_r\ d\ k_r \quad \ldots\ldots\ldots\ldots \quad (12)$$

If the supports are at the same level, the terms containing h in the above equations become zero; the equations remain the same in every other particular.

We have now deduced general equations by which we can determine the bending moments, shears and deflections for any continuous girder. In the next chapters we will give numerous examples or special cases illustrating the application of the formulas.

II.

SUPPORTED GIRDERS.

CASE I.

A simple girder resting upon two supports—

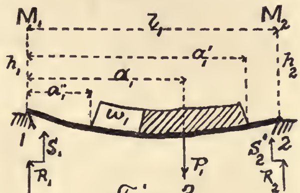


Fig. 3.

(*a*) ***E*** *alone constant.*

From ($\boldsymbol{A}$) we have for $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$

$M_1 = M_2 = 0$ . . . . (103)

From ($\boldsymbol{B}$) and ($\boldsymbol{C}$)

$S_1 = Q_1$ . . . . (104)

$S_2' = Q_1'$ . . . . (105)

Or

$S_1 = \Sigma\ P_1\ (1 - k_1)$ . . *for concentrated loads* . . . . . . (106)

$S_2' = \Sigma\ P_1\ k_1$ . . *for concentrated loads* . . . . (107)

$S_1 = w_1\ l_1\ \left\{k_1 - \frac{k_1^2}{2}\right\}\ \begin{matrix} a_1' = k_1\ l_1 \\ a_1'' = k_1\ l_1 \end{matrix}$ . . . *for any uniform load* . . . (108)

$S_2' = w_1\ l_1\ \left\{\frac{k_1^2}{2}\right\}\ \begin{matrix} a_1' = k_1\ l_1 \\ a_1'' = k_1\ l_1 \end{matrix}$ . . . . *for any uniform load* . . . . (109)

$S_1 = \frac{1}{2}\ w_1\ l_1$ . . . *for uniform load over all* . . . . . . (110)

$$S_2' = \frac{1}{2} w_1 l_1 \text{ . . . . for uniform load over all . . . . } \qquad (111)$$

From ($\boldsymbol{D}$) the bending moment at any section x_1 is

$$\overset{1}{M_x} = S_1 x_1 - L_1 \text{ . . . . } \qquad (112)$$

Substituting the values of S_1 *and* L_1 we have,

For concentrated loads—

$$\overset{1}{M_x} = \Sigma P_1 (1 - k_1) x_1 - \Sigma P_1 (x_1 - a_1) \text{ . . . . } a \leqq x \text{ . . . . } \qquad (113)$$

For any uniform load—

$$\overset{1}{M_x} = x_1 w_1 l_1 \left\{ k_1 - \frac{k_1^2}{2} \right\}_{a_1''}^{a_1'} - w_1 \left\{ a_1 \left(x_1 - \frac{a_1}{2} \right) \right\}_{a_1 = a_1''}^{a_1 = a_1'} \; a_1 \leqq x_1 \qquad (114)$$

For uniform load over all—

$$\overset{1}{M_x} = \frac{1}{2} w_1 l_1 x_1 - \frac{1}{2} w_1 x_1^2 = \frac{1}{2} w_1 x_1 (l_1 - x_1) \text{ . . . . } \qquad (115)$$

For a uniform load over all, the moment at the center of the girder becomes,

$$M_c = \frac{1}{2} w_1 \frac{l_1^2}{2} - \frac{1}{2} w_1 \frac{l_1^2}{4} = \frac{1}{8} w_1 l_1^2 \text{ . . . . } \qquad (116)$$

The well known formula for this case. We see, therefore, that our formulas are perfectly exact for the simple girder, and also that a variable moment of inertia or difference of level of the supports does not effect the values of the bending moments or the shears.

From ($\boldsymbol{E}$), we have,

$$y_1 = h_1 + t_1 x_1 + \frac{1}{6 E \overset{1}{I_x}} \Big(S_1 x_1^3 - \Sigma P_1 (x_1 - a_1)^3 \Big)$$

$$+ \frac{1}{6 E \overset{1}{I_x}} \sum_{v = x_1}^{v = 1} \left(\frac{\overset{1}{I_x}}{\overset{1}{I_{v-1}}} - \frac{\overset{1}{I_x}}{\overset{1}{I_v}} \right) \Big\{ S_1 e_v^2 (3 x_1 - 2 e_v)$$

$$- \Sigma P_1 (e_v - a_1)^3 - 3 (x_1 - e_v) \Sigma P_1 (e_v - a_1)^2 \Big\} \qquad (117)$$

$$t_r = \frac{h_2 - h_1}{l_1} \quad \ldots \ldots \ldots \ldots \quad (118)$$

(*b*) ***E*** *and* ***I*** *constant.*

The moment and shear equations are the same as in (***a***).

$$y_1 = h_1 + t_1 x_1 + \frac{1}{6\ E\ I} \left\{ S_1\ x_1^3 - \Sigma\ P_1\ (x_1 - a_1)^3 \right\} \quad \ldots \quad (119)$$

If the origin is taken at one of the supports, the corresponding value of ***h***, will, of course, equal zero.

CASE II.

A beam continuous over three supports—

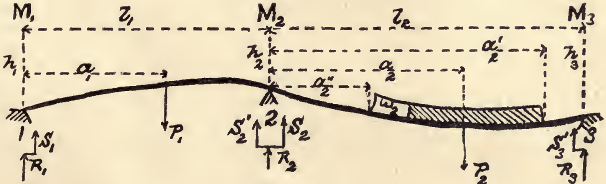


Fig. 4.

(*a*) ***E*** *alone, constant.*

From (*A*),

$$M_1 = M_3 = O \quad \ldots \ldots \ldots \ldots \quad (120)$$

$$M_2 = \frac{-c_2}{d_3\ \beta_1''} \left\{ (A_2 + Y_2 + X_2' + X_1'')\ d_2 + B_2\ d_1 \right\}$$
$$+ \frac{-d_2}{c_3\ \beta_2} \left\{ (A_1 + Y_1 + X_1')\ c_1 + B_1\ c_2 \right\} \quad \ldots \quad (121)$$

$$c_1 = o. \quad c_2 = 1. \quad c_3 = \frac{2\ \beta_2'}{-\beta_2} \quad \ldots \ldots \ldots \quad (122)$$

$$d_1 = o. \quad d_2 = 1. \quad d_3 = -\ \frac{2\ \beta_2'}{\beta_1''} \quad \ldots \ldots \ldots \quad (123)$$

Substituting (*122*) and (*123*) in (*121*), it reduces to,

$$M_2 = \frac{1}{2\,\beta_2'}\ (A_2 + Y_2 + X_2' + X_1'' + B_1) \quad \ldots \ldots \ldots \quad (124)$$

$$X_2' = H_2\, \theta_1 - F_2'\, \Sigma\, P_2\, l_2\, (1-k_2)\, \theta_1 \qquad (125)$$

$$X_1'' = -H_1'\, \theta_2 - 2\, F_1''\, \Sigma\, P_1\, l_1\, (1-k_1)\, \theta_2 \qquad (126)$$

$$H_2 = \sum_{v=l_2}^{v=1} \overset{2}{\underset{v}{\triangle}} \left\{ \frac{\Sigma\, P_2\, (e_v - a_2)^3}{l_2} + \frac{3\,(l_2 - e_v)}{l_2}\, \Sigma\, P_2\, (e_v - a_2)^2 \right\} \qquad (127)$$

$$H_1' = \sum_{v=l_1}^{v=1} \overset{1}{\underset{v}{\triangle}} \left\{ \frac{\Sigma\, P_1\, (e_v - a_1)^3}{l_1} - \frac{3\, e_v}{l_1}\, \Sigma\, P_1\, (e_v - a_1)^2 \right\} \qquad (128)$$

$$\overset{2}{\underset{v}{\triangle}} = \frac{\overset{2}{I_l}}{\overset{2}{I}_{v-1}} - \frac{\overset{2}{I_l}}{\overset{2}{I_v}} \qquad (129)$$

$$\overset{1}{\underset{v}{\triangle}} = \frac{\overset{1}{I_l}}{\overset{1}{I}_{v-1}} - \frac{\overset{1}{I_l}}{\overset{1}{I_v}} \qquad (130)$$

$$F_2' = \sum_{v=l_2}^{v=1} \overset{2}{\underset{v}{\triangle}}\, e_v^2 \left\{ \frac{3}{l_2} - \frac{2\, e_v}{l_2^2} \right\} \qquad (131)$$

$$F_1'' = \sum_{v=l_1}^{v=1} \overset{1}{\underset{v}{\triangle}}\, \frac{e_v^3}{l_1^2} \qquad (132)$$

$$F_2 = \sum_{v=l_2}^{v=1} \overset{2}{\underset{v}{\triangle}} \left\{ \frac{l_2^3 - (l_2 - e_v)^3}{l_2^2} \right\} \qquad (133)$$

$$\beta_2' = 6\, E\, \overset{2}{I_l}\, (l_1 + F_1'') + 6\, E\, \overset{1}{I_l}\, (l_2 + F_2) \qquad (133_a)$$

$$Y_2 = -3\, 6\, E^2\, \overset{2}{I_l}\, \overset{1}{I_l} \left\{ \frac{h_2 - h_1}{l_1} + \frac{h_2 - h_3}{l_2} \right\} \qquad (134)$$

A_2 *and* B_1

Concentrated loads—

$$A_2 = -\Sigma P_2\, l_2^2\, (2\,k_2 - 3\,k_2^2 + k_2^3)\; 6\,E\,\overset{1}{I}_l \quad \ldots\ldots \quad (135)$$

$$B_1 = -\Sigma P_1\, l_1^2\, (k_1 - k_1^3)\; 6\,E\,\overset{2}{I}_l \quad \ldots\ldots \quad (136)$$

Partial uniform loads—

$$A_2 = -w_2\, l_2^3 \left\{ k_2^2 - k_2^3 + \frac{k_2^4}{4} \right\} \Bigg|_{a''_2 = k_2\, l_2}^{a'_2 = k_2\, l_2} 6\,E\,\overset{1}{I}_l \quad \ldots\ldots \quad (137)$$

$$B_1 = -w_1\, l_1^3 \left\{ \frac{2\;k_1^2 - k_1^4}{4} \right\} \Bigg|_{a''_1 = k_1\, l_1}^{a'_1 = k_1\, l_1} 6\,E\,\overset{2}{I}_l \quad \ldots\ldots \quad (138)$$

Uniform load over all—

$$A_2 = -\frac{1}{4}\, w_2\, l_2^3\; 6\,E\,\overset{1}{I}_l \quad \ldots\ldots \quad (139)$$

$$B_1 = -\frac{1}{4}\, w_1\, l_1^3\; 6\,E\,\overset{2}{I}_l \quad \ldots\ldots \quad (140)$$

$$\Sigma\, P_r = \int_{a_r = a''_r}^{a_r = a'_r} w_r\, d\,a_r = \int_{a''_r = k_r\, l_r}^{a'_r = k_r\, l_r} w_r\, l_r\, d\,k_r \quad \ldots\ldots \quad (12)$$

Shears and intermediate bending moments—

For shears and the intermediate bending moments, apply the general equations (***B***) (***c***) and (***D***), and for deflection use equations (***E***) and (***45***).

(*b*) ***E*** *and* ***I*** *constant.*

In this case, we have from the equations under (***a***),

$$M_1 = M_3 = o \quad \ldots\ldots \quad (141)$$

$$M_2 = \frac{Y_2 + A_2 + B_1}{2\,(l_1 + l_2)} \quad \ldots \quad (142)$$

In which,

$$Y_2 = -\,6\,E\,I\left\{\frac{h_2 - h_1}{l_1} + \frac{h_2 - h_3}{l_2}\right\} \quad \ldots \quad (143)$$

A_2 and B_1

Concentrated loads—

$$A_2 = -\,\Sigma\,P_2\,l_2^3\,(2\,k_2 - 3\,k_2^2 + k_2^3) \quad \ldots \quad (144)$$

$$B_1 = -\,\Sigma\,P_1\,l_1^2\,(k_1 - k_1^3) \quad \ldots \quad (145)$$

Uniform load over all—

$$A_2 = -\,w_2\,l_2^3\left\{k_2^2 - k_2^3 + \frac{k_2^4}{4}\right\}_{a''_2 = k_2\,l_2}^{a'_2 = k_2\,l_2} \quad \ldots \quad (146)$$

$$B_1 = -\,w_1\,l_1^3\left\{\frac{2\,k_1^2 - k_1^4}{4}\right\}_{a''_1 = k_1\,l_1}^{a'_1 = k_1\,l_1} \quad \ldots \quad (147)$$

Uniform load over all—

$$A_2 = -\,\frac{1}{4}\,w_2\,l_2^3 \quad \ldots \quad (148)$$

$$B_1 = -\,\frac{1}{4}\,w_1\,l_1^3 \quad \ldots \quad (149)$$

As this is a case much used in practice, we will give expressions for the bending moments in terms of the spans and the loads.

Concentrated loads—

$$M_2 = \frac{Y_2 - \Sigma\,P_2\,l_2^2\,K_2 - \Sigma\,P_1\,l_1^2\,K'_1}{2\,(l_1 + l_2)} \quad \ldots \quad (150)$$

In which,

$$K_2 = 2\,k_2 - 3\,k_2^2 + k_2^3 \quad \ldots \quad (151)$$

And

$$K'_1 = k_1 - k_1^3 \quad \ldots \quad (152)$$

From ($\boldsymbol{B}$) and ($\boldsymbol{C}$),

$$S_1 = \frac{Y_2 - \Sigma\, P_2\, l_2^2\, K_2 - \Sigma\, P_1\, l_1\, K_1'}{2\, l_1\, (l_1 + l_2)} + \Sigma\, P_1\, (1 - k_1) = R_1 \quad . \quad . \quad (153)$$

$$\left.\begin{array}{l} S_2' = \dfrac{-Y_2 + \Sigma\, P_2\, l_2^2\, K_2 + \Sigma P_1\, l_1^2\, K_1'}{2\, l_1\, (l_1 + l_2)} + \Sigma\, P_1\, k_1 \quad . \quad . \quad (154) \\[2ex] S_2 = \dfrac{-Y_2 + \Sigma P_2\, l_2^2\, K_2 + \Sigma\, P_1\, l_1^2\, K_1'}{2\, l_2\, (l_1 + l_2)} + \Sigma\, P_2\, (1 - k_2) \quad . \quad . \quad (155) \end{array}\right\} = R_2$$

Or,

$$R_2 = \frac{-Y_2 + \Sigma P_2\, l_2^2\, K_2 + \Sigma\, P_1\, l_1^2\, K_1'}{2\, l_1\, l_2} + \Sigma\, P_1\, k_1 + \Sigma P_2 (1 - k_2) (156)$$

$$S_3' = \frac{Y_2 - \Sigma\, P_2\, l_2^2\, K_2 - \Sigma\, P_1\, l_1^2\, K_1'}{2\, l_2\, (l_1 + l_2)} + \Sigma\, P_2\, k_2 = R_3 \quad . \quad . \quad . \quad . \quad . \quad (157)$$

Uniform loads over each span—

The above equations at once reduce to

$$M_2 = \frac{Y_2 - \frac{1}{4}\, w_2\, l_2^3 - \frac{1}{4}\, w_1\, l_1^3}{2\, (l_1 + l_2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (158)$$

$$S_1 = \frac{Y_2 - \frac{1}{4} w_2\, l_2^3 - \frac{1}{4}\, w_1\, l_1^3}{2\, l_1\, (l_1 + l_2)} + \frac{1}{2}\, w_1\, l_1 = R_1 \quad . \quad . \quad . \quad . \quad (159)$$

$$S_2' = \frac{-Y_2 + \frac{1}{4}\, w_2\, l_2^3 + \frac{1}{4}\, w_1\, l_1^3}{2\, l_1\, (l_1 + l_2)} + \frac{1}{2}\, w_1\, l_1 \quad . \quad . \quad . \quad . \quad . \quad (160)$$

$$S_2 = \frac{-Y_2 + \frac{1}{4}\, w_2\, l_2^3 + \frac{1}{4}\, w_1\, l_1^3}{2\, l_2\, (l_1 + l_2)} + \frac{1}{2} w_2\, l_2 \quad . \quad . \quad . \quad . \quad . \quad (161)$$

Or,

$$R_2 = \frac{-Y_2 + \frac{1}{4}\, w_2 l_2^3 + \frac{1}{4}\, w_1 l_1^3}{2\, l_1\, l_2} + \frac{1}{2}\, (w_1\, l_1 + w_2\, l_2) \quad . \quad . \quad . \quad (162)$$

$$S_3' = \frac{Y_2 - \frac{1}{4} w_2 l_2^3 - \frac{1}{4} w_1 l_1^3}{2 l_2 (l_1 + l_2)} + \frac{1}{2} w_2 l_2 = R_3 \quad \ldots \ldots \quad (163)$$

(c) ***E, I** and **h** constant.*

In this case, the formulas are the same as in case (***b***) with the term $Y_2 = O$.

(d) ***E, I, h** and **l** constant.*

Making $l_1 = l_2$, the equations of case (***b***) reduce to

Concentrated loads—

$$M_2 = \frac{-\Sigma P_2 l_2 K_2 - \Sigma P_1 l_1 K_1'}{4} \quad \ldots \ldots \quad (164)$$

$$S_1 = \frac{-\Sigma P_2 K_2 - \Sigma P_1 K_1'}{4} + \Sigma P_1 (1 - k_1) = R_1 \quad \ldots \ldots \quad (165)$$

$$\left.\begin{array}{ll} S_2' = \frac{+\Sigma P_2 K_2 + \Sigma P_1 K_1'}{4} + \Sigma P_1 k_1 & \ldots \ldots \quad (166) \\ S_2 = \frac{+\Sigma P_2 K_2 + \Sigma P_1 K_1'}{4} + \Sigma P_2 (1 - k_2) & \ldots \ldots \quad (167) \end{array}\right\} = R_2$$

Or,

$$R_2 = \frac{\Sigma P_2 K_2 + \Sigma P_1 K_1'}{2} + \Sigma P_1 k_1 + \Sigma P_2 (1 - k_2) = R_2 \quad \ldots \quad (168)$$

$$S_3' = \frac{-\Sigma P_2 K_2 - \Sigma P_1 K_1'}{4} + \Sigma P_2 k_2 = R_3 \quad \ldots \ldots \quad (169)$$

Uniform load over all—$w_1 = w_2 = w$

$$M_2 = -\frac{1}{8} w l^2 \quad \ldots \ldots \quad (170)$$

$$S_1 = -\frac{1}{8} w l + \frac{1}{2} w l = \frac{3}{8} w l = R_1 \quad \ldots \ldots \quad (171)$$

$$\left.\begin{array}{ll} S_2' = +\frac{1}{8} w l + \frac{1}{2} w l = \frac{5}{8} w l & \ldots \ldots \quad (172) \\ S_2 = +\frac{1}{8} w l + \frac{1}{2} w l = \frac{5}{8} w l & \ldots \ldots \quad (173) \end{array}\right\} = \frac{10}{8} w l = R_2$$

$$S_3' = -\frac{1}{8} w\, l + \frac{1}{2} w\, l = \frac{3}{8} w\, l = R_3 \quad \ldots\ldots (174)$$

CASE III.

A beam continuous over four supports—

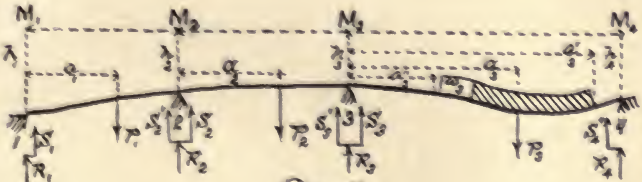


Fig. 5.

(a) ***E*** *alone, constant.*

$$M_1 = M_4 = 0 \quad \ldots\ldots (175)$$

$$M_2 = \frac{-c_2}{d_1\,{}_1\delta_1}\left\{\begin{array}{l}(A_2 + Y_2 + X_2' + X_1'')\, d_3 + B_2\, d_2 \\ (A_3 + Y_3 + X_3' + X_2'')\, d_2 + B_3\, d_1\end{array}\right\}$$

$$+ \frac{-d_3\,{}_1\delta_2''}{c_1\,{}_1\delta_3\,{}_1\delta_2}\left\{(A_1 + Y_1 + X_1' + X_0'')\; c_1 + B_1\, c_2\right\} \quad \ldots (176)$$

Which reduces to the following:

$$M_2 = \frac{-1}{d_1\,{}_1\delta_1}\left\{\begin{array}{l}(A_2 + Y_2 + X_2' + X_1'')\, d_3 + B_2 \\ (A_3 + Y_3 + X_3' + X_2'')\end{array}\right\} + \frac{-d_3}{(c_1\,{}_1\delta_3)}\frac{{}_1\delta_2''}{{}_1\delta_2} B_1 \ldots (177)$$

$$M_3 = \frac{-c_3\,{}_1\delta_2}{(d_1\,{}_1\delta_1)\,{}_1\delta_2''}\left\{(A_3 + Y_3 + X_3' + X_2'')\right\}$$

$$+ \frac{-1}{c_1\,{}_1\delta_3}\left\{\begin{array}{l} + B_1 \\ (A_2 + Y_2 + X_2' + X_1'') + B_2\, c_3\end{array}\right\} \quad \ldots (178)$$

Use general equations for the various terms in the above, and also for shear, intermediate moments and deflection.

(*b*) ***E*** *and* ***I*** *constant.*

Our equations now reduce to

$$M_2 = \frac{(Y_2+A_2+B_1)\,2\,(l_2+l_3)-(A_3+B_2+Y_3)\,l_2}{4\,(l_1+l_2)\,(l_2+l_3)-l_2^2} \quad . \quad . \quad . \quad . \quad (179)$$

$$M_3 = \frac{(Y_3+A_3+B_2)\,2\,(l_1+l_2)-(B_1+A_2+Y_2)\,l_2}{4\,(l_1+l_2)\,(l_2+l_3)-l_2^2} \quad . \quad . \quad . \quad . \quad (180)$$

(*c*) ***E***, ***I*** *and* ***h*** *constant.*

$$M_2 = \frac{2\,(l_2+l_3)\,(A_2+B_1)-l_2\,(A_3+B_2)}{4\,(l_1+l_2)\,(l_2+l_3)-l_2^2} \quad . \quad . \quad . \quad . \quad (181)$$

$$M_3 = \frac{2\,(l_1+l_2)\,(A_3+B_2)-l_2\,(A_2+B_1)}{4\,(l_1+l_2)\,(l_2+l_3)-l_2^2} \quad . \quad . \quad . \quad . \quad (182)$$

(*d*) ***E***, ***I***, ***h*** *and* ***l*** *constant.*

$$M_2 = \frac{4\,(A_2+B_1)-(A_3+B_2)}{15\,l} \quad . \quad . \quad . \quad . \quad (183)$$

$$M_3 = \frac{4\,(A_3+B_2)-(A_2+B_1)}{15\,l} \quad . \quad . \quad . \quad . \quad (184)$$

Uniform loads—

If each span is covered with a uniform load, we have

$$M_2 = \frac{-3\,w_2\,l_2^2-4\,w_1\,l_1^2+w_3\,l_3^2}{60} \quad . \quad . \quad . \quad . \quad (185)$$

$$M_3 = \frac{-3\,w_2\,l_2^2-4\,w_3\,l_3^2+w_1\,l_1^2}{60} \quad . \quad . \quad . \quad . \quad (186)$$

If $w_1=w_2=w_3=w$, and $l_1=l_2=l_3=l$, we obtain, at once, the well known forms,

$$M_2 = -\frac{1}{10} w\,l^2 \quad . \quad . \quad . \quad . \quad (187)$$

$$M_3 = -\frac{1}{10} w\,l^2 \quad . \quad . \quad . \quad . \quad (188)$$

And for the shears we have

$$S_1 = -\frac{1}{10} w\, l + \frac{1}{2} w\, l = \frac{4}{10} w\, l = R_1 \quad \ldots \ldots \quad (189)$$

$$S_2' = +\frac{1}{10} w\, l + \frac{1}{2} w\, l = \frac{6}{10} w\, l \quad \ldots \ldots \quad (190)$$

$$S_2 = +\frac{1}{2} w\, l \qquad = \frac{5}{10} w\, l \quad \ldots \ldots \quad (191)$$

$$R_2 = S_1' + S_2 = \left\{ \frac{6}{10} + \frac{5}{10} \right\} w\, l = \frac{11}{10} w\, l \quad \ldots \ldots \quad (192)$$

$$S_3' = +\frac{1}{2} w\, l = \frac{5}{10} w\, l \quad \ldots \ldots \quad (193)$$

$$S_3 = +\frac{1}{10} w\, l + \frac{1}{2} w\, l = \frac{6}{10} w\, l \quad \ldots \ldots \quad (194)$$

$$R_3 = S_3' + S_3 = \left\{ \frac{5}{10} + \frac{6}{10} \right\} w\, l = \frac{11}{10} w\, l \quad \ldots \ldots \quad (195)$$

$$S_4' = -\frac{1}{10} w\, l + \frac{1}{2} w\, l = \frac{4}{10} w\, l = R_4 \quad \ldots \ldots \quad (196)$$

CASE IV.

"THE TIPPER."

* We will now consider the case of the "Tipper," or a beam continuous over four supports, the second and third supports, resting on a rigid beam, supported, generally, by a single support at its center.

It is evident that if supports **2** and **3** are supported by an unyielding bar, which is supported, in turn, at its center, that the reaction $\boldsymbol{R_2}$ must equal the reaction $\boldsymbol{R_3}$, and also that $+\boldsymbol{h_2} = -\boldsymbol{h_3}$. If the unyielding bar is not supported at its center, $\boldsymbol{R_2}$ and $\boldsymbol{R_3}$ will be inversely proportional to their lever arms around the point of support, and $+\boldsymbol{h_2}$ and $-\boldsymbol{h_3}$ will be directly proportional to the lever arms.

* See "Strains in Framed Structures," page 254, by Prof. DuBois.

The unyielding bar supported at its center—

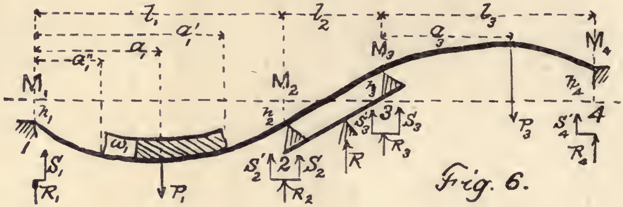


Fig. 6.

(a) ***E** alone, constant.*

From our general equations, by reduction, we obtain,

$$R_2 = \frac{-M_2}{l_1} + \frac{M_3 - M_2}{l_2} + Q'_1 + Q_2 \quad \ldots \ldots \quad (197)$$

$$R_3 = \frac{-M_3}{l_3} + \frac{M_2 - M_3}{l_2} + Q'_2 + Q_3 \quad \ldots \ldots \quad (198)$$

Since $R_2 = R_3$, we have

$$\frac{M_3}{l_3} - \frac{M_2}{l_1} + \frac{2\ M_3 - 2\ M_2}{l_2} + Q'_1 + Q_2 - Q'_2 - Q_3 = 0 \quad \ldots \quad (199)$$

$$Q_2 - Q'_2 = \Sigma\, P_2\, (1 - 2\ k_2) \quad \ldots \ldots \quad (200)$$

Let

$$Q'_1 + Q_2 - Q'_2 - Q_3 = Q \quad \ldots \ldots \quad (201)$$

$$l_1 = n\ l_2 \text{ and } l_3 = m\ l_2 \quad \ldots \ldots \quad (202)$$

Then (199) becomes

$$n\ M_3 - m\ M_2 + 2\ m\ n\ (M_3 - M_2) + m\ n\ Q\ l_2 = 0 \quad \ldots \ldots \quad (203)$$

Or,

$$n\ M_3 - m\ M_2 + 2\ m\ n\ (M_3 - M_2) = -\ m\ n\ Q\ l_2 \quad \ldots \ldots \quad (204)$$

By inspecting the equations under Case III, we see that the moment equations may be placed under the following forms:

$$M_2 = C_2\ Y_2 + C_3\ Y_3 + C_1 \quad \ldots \ldots \quad (205)$$

$$M_3 = C'_2\ Y_2 + C'_3\ Y_3 + C'_1 \quad \ldots \ldots \quad (206)$$

From (205) and (206), we have

$$M_3 - M_2 = (C_2' - C_2)\ Y_2 + (C_3' - C_3)\ Y_3 + (C_1' - C_1) \quad . \quad . \quad . \quad . \quad (207)$$

Substituting (205), (206) and (207) in (204), it becomes

$$\left\{ \begin{array}{l} +n\ C_2'\ Y_2 + n\ C_3'\ Y_3 + n\ C_1' \\ -m\ C_2\ Y_2 - m\ C_3\ Y_3 - m\ C_1 \\ +2\,m\,n\,(C_2' - C_2)\ Y_2 + 2\,m\,n\,(C_3' - C_3)\ Y_3 \\ \qquad\qquad +2\ m\ n\ (C_1' - C_1) \end{array} \right\} = -\,m\,n\ Q\ l_2 \ . \quad (208)$$

Or, by placing V and V' in place of the co-efficients of Y, etc., we have

$$V\ Y_2 + V'\ Y_3 = -\,m\ n\ Q\ l_2 - V'' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (209)$$

$$\text{Now,}\ \ Y_2 = -\,\theta_1\ \theta_2 \left\{ \frac{h_2 - h_1}{n\ l_2} + \frac{h_2 - h_3}{l_2} \right\} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (210)$$

$$Y_3 = -\,\theta_2\ \theta_3 \left\{ \frac{h_3 - h_2}{l_2} + \frac{h_3 - h_4}{m\ l_2} \right\} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (211)$$

But, $h_2 = -h_3$, or, $h_3 = -h_2$; therefore,

$$V\ Y_2 = -\ \theta_1\ \theta_2 \left\{ \frac{-h_1}{n\ l_2} V \right\} - \left\{ \frac{1+2\ n}{n\ l_2} V \right\} \theta_1\ \theta_2\ h_2 \ . \ \ . \quad (212)$$

$$V'\ Y_3 = -\ \theta_2\ \theta_3 \left\{ \frac{-h_4}{m\ l_2} V' \right\} - \left\{ \frac{1+2\ m}{m\ l_2}\ V' \right\}\ \theta_2\ \theta_3\ h_2 \ . \ \ . \quad (213)$$

Hence,

$$V'\ Y_3 + V\ Y_2 = \left[\left\{ \frac{h_1}{n\ l_2} V\ \theta_1 \right\} + \left\{ \frac{h_4}{m\ l_2} V'\,\theta_3 \right\} \right] \theta_2$$

$$-\left[\left\{ \frac{1+2\ n}{n\ l_2}\ V\,\theta_1 \right\} + \left\{ \frac{1+2\ m}{m\ l_2} V'\,\theta_3 \right\} \right] \theta_2\ h_2 \ . \ . \quad (214)$$

Or,

$$V'\ Y_3 + V\ Y_2 = [1]\ \theta_2 - [2]\ \ \theta_2\ \ h_2 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (215)$$

Therefore,

$$[1]\ \theta_2 - [2]\ \theta_2\ h_2 = -\,m\ n\ Q\ l_2 - V'' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (216)$$

And

$$h_2 = \frac{+m\ \ n\ \ Q\ \ l_2 + V'' + [1]\ \theta_2}{[2]\ \ \theta_2} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (217$$

From (217), we can find the value of $h_2 = -h_3$, and then the values of Y_2 and Y_3 from (210) and (211), whence we can determine M_2 and M_3 from (177) and (178).

The easiest method of finding the values of the terms in the equation giving the value of h_2, is to substitute, for any particular case, the values of all the known quantities in the immediately preceding equations, or find the values of the constants C_1, C_2, etc., and in turn, substitute them in the equations containing them. The operations are long and exceedingly tedious, but simple, as an examination of the equations shows.

(b) *E and I constant.*

If the moment of inertia is constant, the deduction of the constants C_1, C_2, etc., becomes much more simple. After h_2 is determined from (217), and Y_2 and Y_3 from (210) and (211), the bending moments are readily obtained from (179) and (180).

If $h_1 = h_4 = o$, as is usually the case, the process becomes still more simple.

(c) *E and I constant.* $h_1 = h_4 = o.$ $l_1 = l_3 = l.$ $l_2 = n\ l.$ $h_2 = -h_3.$

We have, as in (*a*), by a little reduction,

$$n\ (M_3 - M_2) + 2\ (M_3 - M_2) = -n\ l\ Q \quad \text{(218)}$$

Or,

$$(n+2)\ (M_3 - M_2) = -n\ l\ Q \quad \text{(219)}$$

Letting $4\ (l_1 + l_2)\ (l_2 + l_3) - l_2^2 = D\ l$ . . . . (220)

We have, from (179) and (180),

$$M_2 = \frac{(Y_2 + A_2 + B_1)\ 2\ (n+1) - (A_3 + B_2 + Y_3)\ n}{D} \quad \text{(221)}$$

$$M_3 = \frac{(Y_3 + A_3 + B_2)\ 2\ (n+1) - (A_2 + B_1 + Y_2)\ n}{D} \quad \text{(222)}$$

Then,

$$\begin{aligned} D\ (M_3 - M_2) = {} & (Y_3 + A_3 + B_2)\ 2\ (n+1) - (A_2 + B_1 + Y_2)\ n \\ & (Y_3 + A_3 + B_2)\ n \qquad - (A_2 + B_1 + Y_2)\ 2\ (n+1) \end{aligned} \quad \text{(223)}$$

Hence,

$$Y_3\ (3\ n+2)-Y_2\ (3\ n+2)+(A_3+B_2 \\ -A_2-B_1)\ (3\ n+2)=(M_3-M_2)\ D \quad . \quad . \ (224)$$

Substituting (224) in (219), it reduces to

$$(Y_3-Y_2+A_3+B_2-A_2-B_1)\,(3\,n+2)\,(n+2)=-D\,n\,l\,Q \quad . \ (225)$$

Therefore,

$$Y_3-Y_2=\frac{-D\ n\ l\ Q}{(3\ n+2)\ (n+2)}-A_3-B_2+A_2+B_1 \quad . \ . \ . \ . \ (226)$$

Since $h_1=h_4=o$ and $h_2=-h_3$,

$$Y_2=-6\ E\ I\left\{\frac{h_2}{l_2}+\frac{2\ h_2}{l}\right\}=-6\ E\ I\left\{\frac{h_2(1+2\ n)}{n\ l}\right\} \quad . \ (227)$$

$$Y_3=-6\ E\ I\left\{\frac{-h_2}{l}+\frac{-2\ h_2}{l_2}\right\}=+6\ E\ I\left\{\frac{h_2(1+2\,n)}{n\ l}\right\} \quad (228)$$

Therefore,

$$Y_3=-Y_2 \quad . \ (229)$$

And

$$Y_3-Y_2=-2\ Y_2 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (230)$$

Hence,

$$Y_2=\frac{D\ n\ l\ Q}{2\ (3\ n+2)\ (n+2)}+\frac{A_3+B_2-A_2-B_1}{2} \quad . \ . \ . \ . \ . \ . \ (231)$$

But, $D\ l=4\ (l_1+l_2)\ (l_2+l_3)-l_2^2=4\ l_1\ l_3+4\ l_2\ l_3$

$$+4\ l_2\ l_1+4\ l_2^2-l_2^2=4\ l^2+8\ n\ l^2+3\ n^2\ l^2=l^2\ (3\ n^2 \\ +8\ n+4)=l^2\ (3\ n+2)\ (n+2) \quad . \ . \ . \ . \ . \ (232)$$

And

$$Q=Q_1'+Q_2-Q_2'-Q_3 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (201)$$

Substituting (201) and (232) in (231), we have

$$Y_2=\frac{l\,l_2(Q_1'+Q_2-Q_2'-Q_3)+A_3+B_2-A_2-B_1}{2}=-Y_3 \quad . \ . \ (233)$$

(233) completely determines $\boldsymbol{Y_2}$ and $\boldsymbol{Y_3}$, and now the bending moments can be deduced from (179) and (180).

III.

BEAMS WITH FIXED ENDS.

In this chapter we shall consider beams with one or both ends fixed.

CASE I.

A beam fixed at one end and supported at the other—

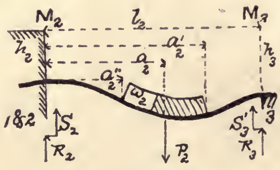


Fig. 7.

For convenience, the left end will be considered as fixed.

This case is, in reality, the same as Case II, in Chapter II, considering $l_1 = o$, $h_1 = h_2$ and $\overset{1}{I_1} = o$, hence, we can use the equations of that case by making the proper changes.

(*a*) ***E*** *alone constant.*

(124) becomes

$$M_2 = \frac{1}{2\,\beta_2'}\,(A_2 + Y_2 + X_2') \quad . \quad . \quad . \quad . \quad . \quad . \quad (234)$$

(134) becomes

$$* \; Y_2 = - 3 \; 6 \; E^2 \overset{2}{I_l} \; \overset{1}{I_l} \; \left\{ \frac{h_2 - h_3}{l_2} \right\} \quad \ldots \ldots \quad (235)$$

By inspecting equations (124) to (134), inclusive, it is seen that we can cancel out $\boldsymbol{6 \; E \; \overset{1}{I_l}}$ from all the terms in (234), hence, we can write

$$* \; Y_2 = - 6 \; E \; \overset{2}{I_l} \frac{h_2 - h_3}{l_2} \quad \ldots \ldots \quad (236)$$

$$\beta_2' = l_2 + F_2 \quad \ldots \ldots \quad (237)$$

$$\overset{2}{\underset{v}{\triangle}} = \frac{\overset{2}{I_l}}{\overset{2}{I}_{v-1}} - \frac{\overset{2}{I_l}}{\overset{2}{I}_v} \quad \ldots \ldots \quad (129)$$

$$F_2' = \sum_{v=l_2}^{v=1} \overset{2}{\underset{v}{\triangle}} \; e_v^2 \left\{ \frac{3}{l_2} - \frac{2 \, e_v}{l_2^2} \right\} \quad \ldots \ldots \quad (131)$$

$$F_2 = \sum_{v=l_2}^{v=1} \overset{2}{\underset{v}{\triangle}} \left\{ \frac{l_2^3 - (l_2 - e_v)^3}{l_2^2} \right\} \quad \ldots \ldots \quad (133)$$

$$H_2 = \sum_{v=l_2}^{v=1} \overset{2}{\underset{v}{\triangle}} \left\{ \frac{\Sigma \; P_2 \, (e_v - a_2)^3}{l_2} + \frac{3 \, (l_2 - e_v)}{l_2} \; \Sigma \; P_2 \, (e_v - a_2)^2 \right\} \quad \ldots \quad (127)$$

$$X_2' = H_2 - F_2' \; P_2 \, l_2 \, (1 - k_2) \quad \ldots \ldots \quad (238)$$

Concentrated loads—

$$A_2 = - \Sigma \; P_2 \, l_2^2 \, (2 \; k_2 - 3 \; k_2^2 + k_2^3) \quad \ldots \ldots \quad (141)$$

---

* In reality, $\overset{1}{I_l} \left\{ \frac{h_1 - h_2}{l_1} \right\} = \overset{1}{I_l} \left\{ \frac{0_1}{0_2} \right\}$ which is indeterminate, but the above form is the only logical one which the expression for $\boldsymbol{Y_2}$ can take.

Partial uniform loads—

$$A_2 = - w_2 \, l_2^3 \left\{ k_2^2 - k_2^3 + \frac{k_2^4}{4} \right\}_{a_2''=k_2 l_2}^{a_2'=k_2 l_2} \quad \ldots \ldots \ldots \ldots (146)$$

Uniform load over all—

$$A_2 = - \frac{1}{4} w_2 \, l_2^3 \quad \ldots \ldots \ldots \ldots (148)$$

Shears and intermediate bending moments—

For shears and the intermediate bending moments, apply the general equations (***B***), (***C***) and (***D***), and for deflection, use equations (***E***) and (***45***).

(*b*) ***E*** *and* ***I*** *constant.*

For this case, (129), (131), (133), (127) and (238) become zero, and (237) reduces to

$$\beta_2' = l_2 \quad \ldots \ldots \ldots \ldots (239)$$

And we have from (234) or (142)

$$M_2 = \frac{Y_2 + A_2}{2 \, l_2} \quad \ldots \ldots \ldots \ldots (240)$$

In which the values of Y_2 and A_2 are given by (236), (144), (146) or (148).

As this is a case quite likely to occur in practice, we will give expressions for the bending moments and shears in terms of the loads and span.

Concentrated loads—

Substituting in (144) in (240), or from (150), we have

$$M_2 = \frac{Y_2 - \Sigma \, P_2 \, l_2^2 \, K_2}{2 \, l_2} \quad \ldots \ldots \ldots \ldots (241)$$

In which $K_2 = 2 \, k_2 - 3 \, k_2^2 + k_2^3$.

From (***B***) and (***C***), or (155),

$$S_2 = \frac{-Y_2 + \Sigma \, P_2 \, l_2^2 \, K_2}{2 \, l_2^2} + \Sigma \, P_2 \, (1 - k_2) = R_2 \quad \ldots \ldots (242)$$

Also, (see 157).

$$S_3' = \frac{Y_2 - \Sigma P_2 l_2^2 K_2}{2 l_2^2} + \Sigma P_2 k_2 = R_3 \quad \ldots \ldots \quad (243)$$

Uniform load over all—

The above equations at once reduce to

$$M_2 = \frac{Y_2 - \frac{1}{4} w_2 l_2^3}{2 l_2} \quad \ldots \ldots \quad (244)$$

$$S_2 = \frac{-Y_2 + \frac{1}{4} w_2 l_2^3}{2 l_2^2} + \frac{1}{2} w_2 l_2 = R_2 \quad \ldots \ldots \quad (245)$$

$$S_3' = \frac{+Y_2 - \frac{1}{4} w_2 l_2^3}{2 l_2^2} + \frac{1}{2} w_2 l_2 = R_3 \quad \ldots \ldots \quad (246)$$

(*c*) ***E***, ***I*** *and* ***h*** *constant.*

In this case the formulas are the same as in case (*b*) with the term $Y_2 = 0$.

The formulas are:

Concentrated loads—

$$M_2 = \frac{-\Sigma P_2 l_2 K_2}{2} \quad \ldots \ldots \quad (247)$$

$$S_2 = \frac{\Sigma P_2 K_2}{2} + P_2 (1 - k_2) = R_2 \quad \ldots \ldots \quad (248)$$

$$S_3' = \frac{-\Sigma P_2 K_2}{2} + \Sigma P_2 k_2 = R_3 \quad \ldots \ldots \quad (249)$$

Uniform load over all—

(244) becomes

$$M_2 = -\frac{1}{8} w_2 l_2^2 \quad \ldots \ldots \quad (250)$$

$$S_2 = \frac{1}{8} w_2 l_2 + \frac{1}{2} w_2 l_2 = \frac{5}{8} w_2 l_2 = R_2 \quad \ldots \ldots \quad (251)$$

$$S_3' = -\frac{1}{8} w_2 l_2 + \frac{1}{2} w_2 l_2 = \frac{3}{8} w_2 l_2 = R_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (252)$$

From (18), we have

$$\overset{2}{M_x} = \frac{1}{8} w_2 (5 l_2 x_2 - l_2^2 - 4 x_2^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (253)$$

$$Max. \overset{2}{\underset{x}{M}} = -\frac{1}{8} w_2 l_2^2 = M_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (254)$$

From ($\boldsymbol{E}_i$) and ($\boldsymbol{4\ 5\ a}$), we obtain, if $h_2 = h_3 = 0$,

$$y_2 = -\frac{1}{48 E I} w_2 x_2^2 (3 l_2^2 - 5 l_2 x_2 + 2 x_2^2) \quad . \quad . \quad x_2 = a_2' \quad . \quad . \quad (255)$$

Special case—

A single concentrated load at the centre of the beam.

By applying Table I, (250) becomes at once

$$M_2 = -P_2 l_2\ 0.1875 = -\frac{3}{16} P_2 l_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (256)$$

$$S_2 = \frac{3}{16} P_2 + \frac{1}{2} P_2 = \frac{11}{16} P_2 = R_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (257)$$

$$S_3' = -\frac{3}{16} P_2 + \frac{1}{2} P_2 = \frac{5}{16} P_2 = R_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (258)$$

$$\overset{2}{M_x} = -\frac{3}{16} P_2 l_2 + \frac{11}{16} P_2 x_2 - P_2 \left\{ x_2 - \frac{1}{2} l_2 \right\} \quad x_2 \geq a_2 = \frac{1}{2} l_2$$

Or,

$$\overset{2}{M_x} = \frac{5}{16} P_2 (l_2 - x_2) \quad . \quad . \quad . \quad x_2 > \frac{1}{2} l_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (259)$$

$$\overset{2}{M_x} = \frac{1}{16} P_2 (11 x_2 - 3 l_2) \quad . \quad . \quad . \quad x_2 < \frac{1}{2} l_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (260)$$

The maximum moment occurs when $x_2 = \frac{1}{2} l_2$, or,

$$Max. \overset{2}{M_x} = \frac{5}{32} P_2 l_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (261)$$

From $(\boldsymbol{E}_1)$, if $h_2 = h_3 = 0$,

$$y_2 = \frac{1}{96\ E\ I}\ P_2\ (15\ l_2\ x_2^2 - 5\ x_2^3$$

$$-12\ l_2^2\ x_2 + 2\ l_2^3)\ .\ .\ x_2 > \frac{1}{2}\ l_2\ .\ .\ (262)$$

And

$$y_2 = \frac{1}{96\ E\ I}\ P_2\ x_2^2\ (11\ x_2 - 9\ l_2)\ .\ .\ .\ x_2 < \frac{1}{2}\ l\ .\ .\ .\ .\ (263)$$

CASE II.

A beam fixed at both ends—

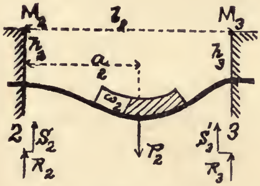


Fig. 8.

This is the same as Case II, Chapter II, with $h_1 = h_2$, $h_4 = h_3$, $l_1 = o$, $l_3 = o$, and $\overset{1}{I_t} = o$, and $\overset{3}{I_t} = o$.

(*a*) ***E*** *alone constant.*

From (177), we have

$$M_2 = \frac{-1}{d_4\ \beta_1''} \left\{ \begin{matrix} (A_2 + Y_2 + X_2')\ d_3 + B_2 \\ (\quad + Y_3 + X_2'') \end{matrix} \right\}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (264)$$

From (178),

$$M_3 = \frac{-c_3\ \beta_2}{(d_4\ \beta_1'')\ \beta_2''} \left\{ + Y_3 + X_2'' \right\}$$

$$+ \frac{-1}{c_4\ \beta_3} \left\{ (A_2 + Y_2 + X_2') + B_2\ c_3 \right\}\ .\ .\ (265)$$

(264) reduces to

$$M_2 = \frac{-\beta_2''}{4\,\beta_2'\,\beta_3' - \beta_2\,\beta_2''}\left\{ \begin{matrix} (A_2 + Y_2 + X_2')\dfrac{-2\,\beta_3'}{\beta_2''} \\ (B_2 + Y_3 + X_2'') \end{matrix} \right\} \quad . \; . \; . \; . \quad (266)$$

And (178) becomes

$$M_3 = \frac{2\,\beta_2'}{4\,\beta_2'\,\beta_3' - \beta_2\,\beta_2''}\left\{ Y_3 + X_2'' \right\}$$

$$+ \frac{-\beta_2}{4\,\beta_2'\,\beta_3' - \beta_2\,\beta_2''}\left\{ (A_2 + Y_2 + X_2') + (B_2)\frac{-2\,\beta_2'}{\beta_2} \right\} \quad . \quad (267)$$

In which, after dividing by $\theta_1 = \theta = \theta_3$,

$$\beta_2' = l_2 + F_2 \quad . \; . \quad (268)$$

$$\beta_3' = l_2 + F_2'' \quad . \; . \; . \quad \text{The values of } F \text{ can be found from (a) (b) (c) and (d)} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (269)$$

$$\beta_2 = l_2 + F_2' \quad . \; . \quad (270)$$

$$\beta_2'' = l_2 + F_2' \quad . \; . \quad (271)$$

$$* \quad Y_2 = -\theta_2 \left\{ \frac{h_2 - h_3}{l_2} \right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (272)$$

$$* \quad Y_3 = -\theta_2 \left\{ \frac{h_3 - h_2}{l_2} \right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (273)$$

The values of the remaining terms are easily found from the general equations, remembering that θ_1 or θ_2 has been cancelled out of each term.

(b) *E and I constant.*

From (266) or (179), we have

$$M_2 = \frac{(Y_2 + A_2)\,2\,l_2 - (B_2 + Y_3)\,l_2}{3\,l_2^2} = \frac{2\,(Y_2 + A_2) - (B_2 + Y_3)}{3\,l_2} \quad . \; . \quad (274)$$

From (267) or (180), we obtain

$$M_3 = \frac{(Y_3 + B_2)\,2\,l_2 - (A_2 + Y_2)\,l_2}{3\,l_2^2} = \frac{2\,(Y_3 + B_2) - (A_2 + Y_2)}{3\,l_2} \quad . \; . \quad (275)$$

In which the terms have the values given under (*a*).

---

* See note under Equation (235).

(c) ***E*, *I* and *h* constant.**

(274) becomes

$$M_2 = \frac{2\,A_2 - B_2}{3\,l_2} \quad \ldots \ldots \ldots \ldots \quad (276)$$

(275) becomes

$$M_3 = \frac{2\,B_2 - A_2}{3\,l_2} \quad \ldots \ldots \ldots \ldots \quad (277)$$

For concentrated loads we can write

$$M_2 = \frac{-2\,\Sigma P_2\,l_2\,K_2 + \Sigma P_2\,l_2\,K_2'}{3} \quad \ldots \ldots \ldots \quad (278)$$

And

$$M_3 = \frac{-2\,\Sigma P_2\,l_2\,K_2' + \Sigma P_2\,l_2\,K_2}{3} \quad \ldots \ldots \ldots \quad (279)$$

Uniform load over all—

$$M_2 = -\frac{1}{12}\,w_2\,l_2^2 = M_3 \quad \ldots \ldots \ldots \ldots \quad (280)$$

$$S_2 = \frac{1}{2}\,l_2\,w_2 = S_3' \quad \ldots \ldots \ldots \ldots \quad (281)$$

$$\overset{2}{M_x} = w_2 \left\{ \frac{l_2\,x_2}{2} - \frac{x_2^2}{2} - \frac{l_2^2}{12} \right\} \quad \ldots \ldots \ldots \quad (282)$$

If $x = 0$

$$Max.\quad \overset{2}{M_x} = -\frac{1}{12}\,w_2\,l_2^2 = M_2 = M_3 \quad \ldots \ldots \ldots \quad (283)$$

If $h_2 = h_3 = 0$,

$$y_2 = \frac{-1}{24\,E\,I}\,w_2\,x_2^2\,(l_2^2 - 2\,l_2\,x_2 + x_2^2) \quad \ldots \quad x_2 = a_2' \quad \ldots \quad (284)$$

A single load in the centre of the beam—

By using Table I, we have at once from (278) and (279),

$$M_2 = M_3 = -0.125\,P_2\,l_2 = -\frac{1}{8}\,P_2\,l_2 \quad \ldots \ldots \ldots \quad (285)$$

$$S_2 = S_3' = \frac{1}{2}\,P_2 \quad \ldots \ldots \ldots \ldots \quad (286)$$

$$\overset{2}{M}_x = \frac{1}{8}\ P_2\,(3\,l_2 - 4\,x_2)\ .\ .\ x_2 > a_2 = \frac{1}{2}\ l_2\ .\ .\ .\ .\ .\ .\ .\ (287)$$

Also,

$$\overset{2}{M}_x = \frac{1}{8}\ P_2\ (4\,x_2 - l_2)\ .\ .\ .\ x_2 < \frac{1}{2}\ l_2\ .\ .\ .\ .\ .\ .\ .\ (288)$$

If $x_2 = 0$, or $\frac{1}{2}\ l_2$.

$$Max.\quad \overset{2}{M}_x = -\ \frac{1}{8}\ w_2\,l_2, \text{ or } +\ \frac{1}{8}\ w_2\,l_2\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (289)$$

$$y_2 = \frac{-1}{48\ E\ I}\ P_2\,(4\,x_2^3 + 6\,l_2^2\,x_2 - 9\,l_2\,x_2^2 - l_2^3)\ .\ .\ x_2 > a_2 = \frac{1}{2}\ l_2\ .\ (290)$$

And

$$y_2 = \frac{-1}{48\ E\ I}\ P_2\,x_2^2\ (3\,l_2 - 4\,x_2)\ .\ .\ x_2 < a_2 = \frac{1}{2}\ l_2\ .\ .\ (291)$$

CASE III.

A beam fixed at one end, and unsupported at the other—

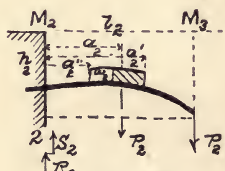


Fig. 7a.

As the right end of the beam is unsupported, there can be no reaction $S_3' = R_3$, therefore, by (11),

Concentrated loads—

$S_3' = \dfrac{M_2 - M_3}{l_2} + \Sigma\ P_2\ k_2 = 0$, and since $M_3 = 0$, we have

$$M_2 = -\Sigma P_2\, l_2\, k_2 \quad . \quad . \quad . \quad S_2 = \Sigma P_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (292)$$

Showing that the moment of inertia does not enter into the expression for the bending moment.

Any uniform load—

$$M_2 = -\frac{1}{2}\, w_2\, (a_2' - a_2'')\, (a_2' + a_2'') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (293)$$

$$S_2 = w_2\, (a_2' - a_2'') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (294)$$

From (8),

$$\overset{2}{M_x} = 0. \qquad x_2 \overline{\gt} a_2$$

And

$$\overset{2}{M_x} = \Sigma P_2\, (x_2 - a_2) \quad . \quad . \quad . \quad x_2 < a_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (295)$$

Also, for any uniform load,

$$\overset{2}{M_x} = w_2\, (a_2' - a_2'') \left\{ x_2 - \frac{a_2' + a_2''}{2} \right\} \quad . \quad . \quad . \quad x_2 \leqq a_2'' \quad . \quad . \quad . \quad . \quad . \qquad (296)$$

For deflection, use the general formulas, if the moment of inertia is variable.

If the moment of inertia is constant, we have, from ($\boldsymbol{E}_1$),

If $h_2 = 0$,

$$y_2 = +\frac{1}{6\, E\, I} \left\{ 3\, M_2\, x_2^2 + \boldsymbol{S}_2\, x_2^3 - \Sigma P_2\, (x_2 - a_2)^3 \right\} \quad . \quad . \quad . \qquad (297)$$

Or, for a single concentrated load,

$$y_2 = \frac{P_2}{6\, E\, I} \left\{ -3\, a_2\, x_2^2 + x_2^3 - (x_2 - a_2)^3 \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (298)$$

And if the load is at the end of the beam, $x_2 = a_2 = l_2$, and (298) becomes

$$y_2 = -\frac{P_2\, l_2^3}{3\, E\, I} \quad . \quad . \quad . \quad \textit{at end of beam} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (299)$$

Also, for a uniform load over all, ($x_2 = a_2 = l_2$),

$$y_2 = \frac{-1}{8\, E\, I}\, w_2\, l_2^4 \quad . \quad . \quad . \quad \textit{at end of beam} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (300)$$

CASE IV.

A beam on two supports, and one end unsupported—

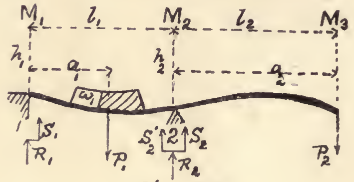


Fig. 7b.

In this case, $\boldsymbol{M_1}$, $\boldsymbol{M_3}$ and $\boldsymbol{S_3}$ equal zero, hence, from (***11***), or (***C***), we have

$$S_3' = \frac{M_2}{l_2} + Q_2' = 0, \text{ or, } M_2 = -Q_2' \, l_2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (301)$$

Which is precisely the same equation we obtained in Case III. The values of $\boldsymbol{Q_2'}$ are given in (95), (97) and (99).

From the general equations (***B***) and (***C***), we obtain

$$S_1 = \frac{M_2}{l_1} + Q_1 = R_1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (302)$$

$$\left.\begin{aligned} S_2' &= \frac{-M_2}{l_1} + Q_1' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (303) \\ S_2 &= \frac{-M_2}{l_2} + Q_2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (304) \end{aligned}\right\} = R_2$$

From (***D***), we obtain

$$\overset{1}{M_x} = S_1 \, x_1 - L_1 = \frac{M_2 \, x_1}{l_1} + Q_1 \, x_1 - L_1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (305)$$

$$\overset{2}{M_x} = M_2 + S_2 \, x_2 - L_2 = M_2 + \frac{-M_2 \, x_2}{l_2} + Q_2 \, x_2 - L_2 \quad . \; . \; . \; . \quad (306)$$

Note that the moment of inertia does not appear in any of the above equations.

For deflection, use the *general equation* (***E***).

CASE V.

A beam on one support, having one end fixed, and the other unsupported—

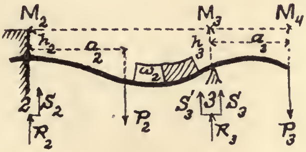


Fig. 8a.

(*a*) ***E*** *alone, constant.*

In this case, as before, $M_4=0$, and $S_4'=0$, hence,

$$M_3 = -Q_3' l_3 \quad \ldots \quad (307)$$

From (63), we have

$$M_2 \beta_2 + 2 M_3 \beta_3' = A_3 + B_2 + Y_3 + X_3' + X_2'' \quad \ldots \quad (308)$$

Substituting (307) in (308), it becomes, after reduction,

$$M_2 = \frac{1}{\beta_2} \left\{ 2 Q_3' l_3 \beta_3' + A_3 + Y_3 + X_3' + B_2 + X_2'' \right\} \quad \ldots \quad (309)$$

(*b*) ***E*** *and* ***I*** *constant.*

(309) becomes

$$M_2 = \frac{1}{l_2} \left\{ 2 Q_3' l_3 (l_2 + l_3) + Y_3 + A_3 + B_2 \right\} \quad \ldots \quad (310)$$

$$M_3 = -Q_3' l_3 \quad \ldots \quad (307)$$

(*c*) ***E***, ***I*** *and* ***h*** *constant.*

$$M_2 = \frac{1}{l_2} \left\{ 2 Q_3' l_3 (l_2 + l_3) + A_3 + B_2) \right\} \quad \ldots \quad (311)$$

$$M_3 = -Q_3' l_3 \quad \ldots \quad (307)$$

The shears, intermediate bending moments and deflections are readily obtained from the general equations.

CASE VI.

A beam on two supports, having neither end supported—

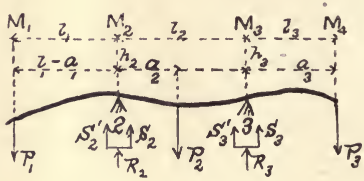


Fig. 8b.

We have, at once, M_1 and M_4 equal zero, and hence, from (***B***),

$$S_1 = \frac{M_2}{l_1} + Q_1 = 0, \text{ or } M_2 = -\, Q_1\, l_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (312)$$

And, from (***C***),

$$S_4' = 0 = \frac{M_3}{l_3} + Q_3', \text{ or } M_3 = -\, Q_3'\, l_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (313)$$

The shears, intermediate bending moments and deflections can now be readily obtained from the general equations.

IV.

* THE POINT OF ZERO MOMENT.

Let us take ($\boldsymbol{D}$).

$$\overset{r}{M_x} = M_r + S_r\, x_r - L_r \quad \ldots\ldots\ldots\ldots \quad (D)$$

In which $\boldsymbol{L}$ is dependent upon the kind of loading in the span $\boldsymbol{r}$. If there is no load in the span $\boldsymbol{r}$, then

$$\overset{r}{M_x} = M_r + S_r\, x_r \quad \ldots\ldots\ldots\ldots \quad (314)$$

Now, if there is a point of zero moment anywhere in the span $\boldsymbol{r}$, we can find its distance from the left support by making (314) equal zero, and solving for x_r; doing this, we obtain

$$x_r = \frac{-M_r}{S_r} \quad \ldots\ldots\ldots\ldots \quad (315)$$

From ($\boldsymbol{B}$),

$$S_r = \frac{M_{r+1} - M_r}{l_r} + (Q_r \text{ in this case} = 0) \quad \ldots\ldots \quad (B)$$

Now,

$$M_m = c_m \frac{\beta_2\, \beta_3 \;.\;.\;.\; \beta_{m-1}}{\beta''_2\, \beta''_3 \;.\;.\;.\; \beta''_{m-1}}\, M_2 \;.\;.\;.\; m < r+1 \quad \ldots\ldots \quad (70)$$

$m < r+1$ indicating that only those loads upon the right of the span are considered.

Substituting ($\boldsymbol{B}$) and (70) in (315), it reduces to

$$x_r = \frac{c_r}{c_r - c_{r+1}\dfrac{\beta_r}{\beta''_r}}\, l_r \;.\;.\;.\; \text{Load on the Right} \quad \ldots\ldots \quad (316)$$

* See "Annales des Ponts et Chaussèes," 1886, Paper No. 40, by M. Collignon.

We also have, for loads on the left,

$$M_m = d_{s-m+2} \frac{\beta''_{s-1}\,\beta''_{s-2}\,.\;.\;.\;\beta''_m}{\beta_{s-1}\,\beta_{s-2}\,.\;.\;.\;\beta_m} M_s \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (71)$$

And, substituting (71) and ($\boldsymbol{B}$) in (315), we obtain

$$x_r = \frac{d_{s-r+2}}{d_{s-r+2} - d_{s-r+1}\,\dfrac{\beta''_{r+1}}{\beta_{r+1}}}\, l_r \quad . \; . \; . \; \text{LOAD ON THE LEFT} \; . \; . \; . \; . \; . \quad (317)$$

(316) and (317) are general equations. Knowing the points of zero moment, we can tell at once what loads must be considered to have a certain effect.

The values of x_r are very easily computed, but they can be constructed graphically as soon as c_r, c_{r+1}, etc., are known. Thus:—

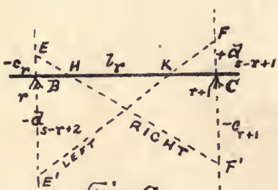


Fig. 9.

Let $\boldsymbol{B\,C}$ represent any unloaded span. Then, $\boldsymbol{B\,H} = x_r$ for $m < r+1$ or a load on the right, and $\boldsymbol{B\,K} = x_r$ for $m > r$ or a load on the left, if $\boldsymbol{E\,B} = c_r$, $\boldsymbol{C\,F'} = -c_{r+1}\,\dfrac{\beta_r}{\beta''_r}$
$\boldsymbol{F\,C} = d_{s-r+1}\dfrac{\beta''_{r+1}}{\beta_{r+1}}$ and $\boldsymbol{B\,E'} = -d_{s-r+2}$.

For, from Fig. 9,

$$B\;H = B\;C\;\frac{E\;B}{E\;B + F'\;C} = \frac{c_r}{c_r + c_{r+1}\,\dfrac{\beta_r}{\beta''_r}}\, l_r = x_r \quad . \; . \; . \; m < r-1$$

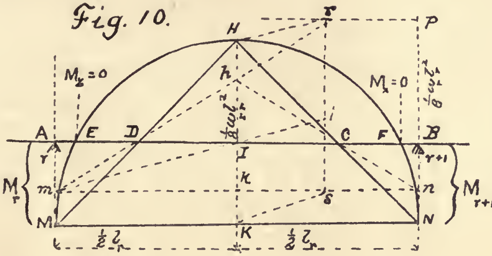
Also,

$$B\,K = B\,C\,\frac{B\,E'}{B\,E'+C\,F} = \frac{d_{s-r+2}}{d_{s-r+2}-d_{s-r+1}\,\dfrac{\beta''_{r+1}}{\beta_{r+1}}}\,l_r = x_r\,.\;\;.\;\;m>r$$

If the span considered is loaded, then the value of x_r must be obtained from (D).

If the supports are level, and the span considered is uniformly loaded, x_r, and also the bending moments, can be found graphically.

Fig. 10.



Considering the span r as discontinuous and uniformly loaded, the bending moment at the centre equals $\frac{1}{8}\,w_r\,l_r^2$. Now, in Fig. 10, we wish to draw the moment parabola, so that $A\,M = M_r$ and $B\,N = M_{r+1}$. We know $H\,K$ must equal $\frac{1}{8}\,w_r\,l_r^2$. Let D and C be the points of zero moment for loads on the right and left of the span r. On the vertical $H\,K$ take any point h and draw $h\,D\,m$ and $h\,C\,n$, and connect m and n. Make $n\,p = \frac{1}{8}\,w_r\,l_r^2$ and draw $p\,r$ parallel to $m\,n$ until it is intersected by $m\,h$, produced in the point r. Draw $r\,H$ and $s\,K$ parallel to $I\,i$, and also $M\,N$ parallel to $m\,n$. Then is $H\,K = \frac{1}{8}\,w_r\,l_r^2$, $A\,M = M_r$, $B\,N = M_{r+1}$ and $A\,E$ and $A\,F = x_r$.

The co-efficients c and d.

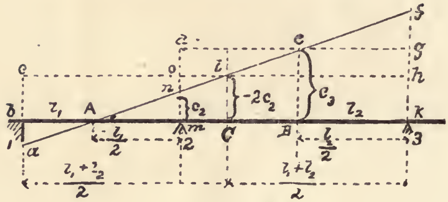


Fig. 11.

The co-efficients c and d can be found graphically, if I is considered as constant.

In Fig. 11, let $b\ m$ and $m\ k$ represent the first and second spans of a continuous girder. Bisect $b\ m$ at A, $m\ k$ at B, and $b\ k$ at C. Make $C\ l = -2\ c_2 = -2$ and draw $a\ A\ l\ e\ f$, a right line passing through A and l, then will $B\ e$ equal, numerically, c_3. Thus:—

$b\ c\ h\ k = -2\ c_2\ (l_1 + l_2) = -2\ (l_1 + l_2)$,

but $b\ c\ h\ k = f\ A\ k - a\ b\ A = f\ n\ m\ k$,

since $d\ e\ n = f\ e\ g$, $f\ n\ m\ k = g\ d\ m\ k = l_2\ (B\ e)$,

Therefore, $B\ e = -2\ \dfrac{l_1 + l_2}{l_2} = c_3$.

d_3 is found in a similar manner from Fig. 12.

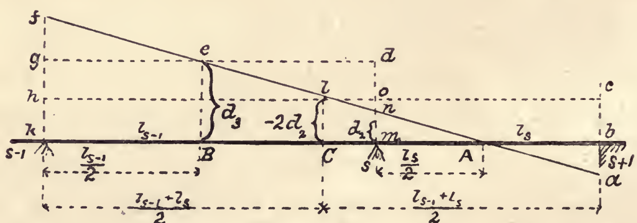


Fig. 12.

$$k\ B = \frac{l_{s-1}}{2},\ bA = \frac{l_s}{2},\ kC = \frac{l_{s-1} + l_s}{2},\ \ C\ l = -2\ d_2 = -2.$$

$b\ c\ h\ k = -2\ d_2\ (l_s + l_{s-1}) = -2\ (l_s + l_{s-1}),$

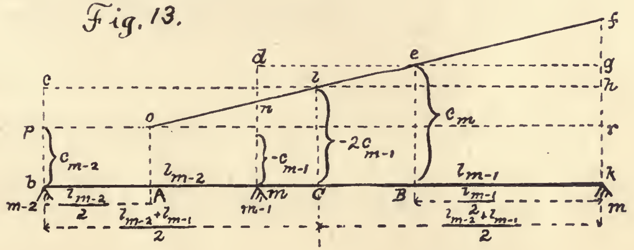
$k\ g\ d\ m = (B\ e)\ l_{s-1}$, but $b\ c\ h\ k = k\ g\ d\ m$,

Therefore,

$$B\ e = -2\ \frac{l_s + l_{s-1}}{l_{s-1}} = d_3.$$

Then, in general, from Fig. 13, we have, if

Fig. 13.



$p\ b = c_{m-2}$ and $l\ C = -2\ c_{m-1}$,

$c\ b\ k\ h = d\ g\ r\ d' + p\ r\ k\ b = d\ g\ k\ m + p\ d'\ m\ b$,

but, $g\ d\ m\ k = (B\ e)\ l_{m-1}$, $p\ d'\ m\ b = c_{m-2}\ l_{m-2}$

and $c\ b\ k\ h = -2\ c_{m-1}\ (l_{m-1} + l_{m-2})$.

Therefore,

$$B\ e = -2\ c_{m-1}\ \frac{l_{m-1} + l_{m-2}}{l_{m-2}} - c_{m-2}\ \frac{l_{m-2}}{l_{m-1}} = c_m.$$

And for d_m we can write

$$B\ e = -2\ d_{m-1}\ \frac{l_{s-m+2} + l_{s-m+3}}{l_{s-m+2}} - d_{m-2}\ \frac{l_{s-m+3}}{l_{s-m+2}} = d_m.$$

V.

APPLICATIONS.

We will now give a complete analysis of a continuous girder, to illustrate, more particularly, the use of Table I.

Ex. 1.—Let Fig. 14 represent a continuous girder of three spans on level supports, and having a constant moment of inertia.

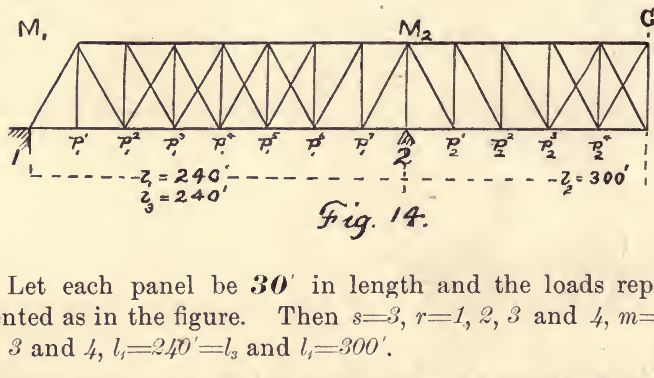


Fig. 14.

Let each panel be ***30'*** in length and the loads represented as in the figure. Then $s=3$, $r=1$, 2, 3 and 4, $m=1$, 2, 3 and 4, $l_1=240'=l_3$ and $l_1=300'$.

First, look up in Table I, the values of the co-efficients $2\,k-3\,k^2+k^3$ and $k-k_3$, for the different values of $\frac{a}{l}=k$.

$k_1=k_3$	$2\,k_1-3\,k_1^2+k_1^3$	$k_1-k_1^3$	k_2	$2k_2-3k_2^2+k_2^3$	$k_2-k_2^3$
			0.100	0.171	0.099
			0.200	0 288	0.192
0.125	0.205,078,125	0.123,046,875	0.300	0.357	0.273
0.250	0.328,125,000	0.234,375,000	0.400	0.384	0.336
0.375	0.380,859,375	0.322,265,625	0.500	0.375	0.375
0.500	0.375,000,000	0.375,000,000	0.600	0.336	0 384
0.625	0.322,265,625	0.380,859,375	0.700	0.273	0.357
0.750	0,234,375,000	0.328,125,000	0.800	0.192	0.288
0.875	0.123,046,875	0.205,078,125	0.900	0.099	0.171

From (A_1), we have $M_1 = 0 = M_4$.

$$M_2 = \frac{-c_2}{d_4\, l_1}\left\{\begin{matrix} A_2\, d_3 + B_1\, d_2 \\ A_3\, d_2 + B_3\, d_1 \end{matrix}\right\} + \frac{-d_3}{c_4\, l_3}\left\{ A_1\, c_1 + B_1\, c_2 \right\}$$

$$M_3 = \frac{-c_3}{d_4\, l_1}\left\{ A_3\, d_2 + B_3\, d_1 \right\} + \frac{-d_2}{c_4\, l_3}\left\{\begin{matrix} A_1\, c_1 + B_1\, c_2 \\ A_2\, c_2 + B_2\, c_3 \end{matrix}\right\}.$$

From (p_1) and (r_1), we find that, since $c_1 = 0$ and $c_2 = 1$, $c_3 = -3.6$ and $c_4 = +14.95$.

Also,

$d_1 = 0$ and $d_2 = 1$, $d_3 = -3.6$ and $d_4 = +14.95$.

Finding the values of A_1, A_2, B_1, B_2, etc., from (i_1) and (j_1), and substituting them, with the values of c and d above, in the moment equations, we obtain,

TABLE a—VALUES OF M_2.

$$\left.\begin{array}{l} -P_1^1\ (0.123{,}046{,}875) \\ -P_1^2\ (0.234{,}375{,}000) \\ -P_1^3\ (0.322{,}265{,}625) \\ -P_1^4\ (0.375{,}000{,}000) \\ -P_1^5\ (0.380{,}859{,}375) \\ -P_1^6\ (0.328{,}125{,}000) \\ -P_1^7\ (0.205{,}078{,}125) \end{array}\right\} 57.7926 = \ldots\ldots \left\{\begin{array}{l} -\ 7.111{,}198\ P_1^1 \\ -\ 13\ 545{,}140\ P_1^2 \\ -\ 18.624{,}568\ P_1^3 \\ -\ 21.672{,}225\ P_1^4 \\ -\ 23\ 010{,}853\ P_1^5 \\ -\ 18.963{,}196\ P_1^6 \\ -\ 11.851{,}998\ P_1^7 \end{array}\right.$$

$$\left.\begin{array}{l} -P_2^1\ (0.171) \\ -P_2^2\ (0.288) \\ -P_2^3\ (0.357) \\ -P_2^4\ (0.384) \\ -P_2^5\ (0.375) \\ -P_2^6\ (0.336) \\ -P_2^7\ (0.273) \\ -P_2^8\ (0.192) \\ -P_2^9\ (0.099) \end{array}\right\} 90.3010 + 25.0836 \left\{\begin{array}{l} +P_2^1\ (0.099) \\ +P_2^2\ (0.192) \\ +P_2^3\ (0.273) \\ +P_2^4\ (0.336) \\ +P_2^5\ (0.375) \\ +P_2^6\ (0.384) \\ +P_2^7\ (0.357) \\ +P_3^8\ (0.288) \\ +P_2^9\ (0.171) \end{array}\right\} = \left\{\begin{array}{l} -\ 12.958{,}194\ P_2^1 \\ -\ 21.190{,}636\ P_2^2 \\ -\ 25.389{,}634\ P_2^3 \\ -\ 26.247{,}494\ P_2^4 \\ -\ 24.456{,}525\ P_2^5 \\ -\ 20.709{,}033\ P_2^6 \\ -\ 15.697{,}327\ P_2^7 \\ -\ 10.113{,}715\ P_2^8 \\ -\ 4.650{,}503\ P_2^9 \end{array}\right.$$

$$\left.\begin{array}{l} +P_3^1\ (0.205{,}078{,}125) \\ +P_3^2\ (0.328{,}125{,}000) \\ +P_3^3\ (0.380{,}859{,}375) \\ +P_3^4\ (0.375.000{,}000) \\ +P_3^5\ (0.322{,}265{,}625) \\ +P_3^6\ (0.234{,}375{,}000) \\ +P_3^7\ (0.123{,}046{,}875) \end{array}\right\} 16.0535 = \ldots\ldots \left\{\begin{array}{l} +\ 3.292{,}221\ P_3^1 \\ +\ 5.267{,}554\ P_3^2 \\ +\ 6.114{,}126\ P_3^3 \\ +\ 6.020{,}061\ P_3^4 \\ +\ 5.173{,}491\ P_3^5 \\ +\ 3.762{,}365\ P_3^6 \\ +\ 1.975{,}333\ P_3^7 \end{array}\right.$$

TABLE a—Continued.

VALUES OF M_3.

$$
\left.\begin{array}{l}
+P_1^1\ (0.123{,}046{,}875)\\
+P_1^2\ (0.234{,}375{,}000)\\
+P_1^3\ (0.322{,}265{,}625)\\
+P_1^4\ (0.375{,}000{,}000)\\
+P_1^5\ (0.380{,}859{,}375)\\
+P_1^6\ (0.328{,}125{,}000)\\
+P_1^7\ (0.205{,}078{,}125)
\end{array}\right\} 16.0535 = \ldots\ldots\ldots \left\{\begin{array}{l}
+1.975{,}333\ P_1^1\\
+3.762{,}365\ P_1^2\\
+5.173{,}491\ P_1^3\\
+6.020{,}061\ P_1^4\\
+6.114{,}126\ P_1^5\\
+5.267{,}554\ P_1^6\\
+3.292{,}221\ P_1^7
\end{array}\right.
$$

$$
\left.\begin{array}{l}
+P_2^1\ (0.171)\\
+P_2^2\ (0.288)\\
+P_2^3\ (0.357)\\
+P_2^4\ (0.384)\\
+P_2^5\ (0.375)\\
+P_2^6\ (0.336)\\
+P_2^7\ (0.273)\\
+P_2^8\ (0.192)\\
+P_2^9\ (0.099)
\end{array}\right\} 25.0836 + 90.3010 \left\{\begin{array}{l}
-P_2^1\ (0.099)\\
-P_2^2\ (0.192)\\
-P_2^3\ (0.273)\\
-P_2^4\ (0.336)\\
-P_2^5\ (0.375)\\
-P_2^6\ (0.384)\\
-P_2^7\ (0.357)\\
-P_2^8\ (0.288)\\
-P_2^9\ (0.171)
\end{array}\right\} = \left\{\begin{array}{l}
-4.650{,}503\ P_2^1\\
-10.113{,}715\ P_2^2\\
-15.697{,}327\ P_2^3\\
-20.709{,}033\ P_2^4\\
-24.456{,}525\ P_2^5\\
-26.247{,}494\ P_2^6\\
-25.389{,}634\ P_2^7\\
-21.190{,}636\ P_2^8\\
-15.958{,}194\ P_2^9
\end{array}\right.
$$

$$
\left.\begin{array}{l}
-P_3^1\ (0.205{,}078{,}125)\\
-P_3^2\ (0.328{,}125{,}000)\\
-P_3^3\ (0.380{,}859{,}375)\\
-P_3^4\ (0.375{,}000{,}000)\\
-P_3^5\ (0.322{,}265{,}625)\\
-P_3^6\ (0.234{,}375{,}000)\\
-P_3^7\ (0.123{,}046{,}875)
\end{array}\right\} 57.7926\ \ldots\ldots\ldots \left\{\begin{array}{l}
-11.851{,}998\ P_3^1\\
-18.963{,}196\ P_3^2\\
-22.010{,}853\ P_3^3\\
-21.672{,}225\ P_3^4\\
-18.624{,}568\ P_3^5\\
-13.545{,}140\ P_3^6\\
-7.111{,}198\ P_3^7
\end{array}\right.
$$

Compare the values of M_2 and M_3, and notice that the co-efficients of P_3^7, P_3^6, etc., of M_3, are the same as the co efficients of P_1^1, P_1^2, etc., of M_2, as they should be, owing to the symmetry of the girder.

TABLE *b*.

The next step is the deduction of S from (B) and (C).

		FIRST SPAN.		SECOND SPAN.		THIRD SPAN.	
		S_1	S'_2	S_2	S'_3	S_3	S'_4
FIRST SPAN.	P_1^1	+0 845,370	+0.154,630	+0.030,288		−0.008,230	
	P_1^2	+0.693,562	+0.306,438	+0.057,691		−0.015,676	
	P_1^3	+0.547,397	+0.452,603	+0.079,327		−0.021,556	
	P_1^4	+0.409,699	+0.590,301	+0.092,307	Same as S_2 with−sign	−0.025,083	Same as S_3 with+sign
	P_1^5	+0.283,288	+0.716,712	+0.093,749		−0.025,475	
	P_1^6	+0.170,986	+0.829,014	+0.080,769		−0.021,948	
	P_1^7	+0.075,616	+0.924,384	+0.050,480		−0.013,717	
SECOND SPAN.	P_2^1	−0.053,992		+0.927,692	+0.072,307	+0.019,377	
	P_2^2	−0.088,294		+0.836,923	+0.163,076	+0.042,140	
	P_2^3	−0.105,790		+0.732,307	+0.267,692	+0.065,405	
	P_2^4	−0.109,364		+0.618,461	+0.381,538	+0.086,287	
	P_2^5	−0.101,902	Same as S_1 with+sign	+0.500,000	+0.500,000	+0.101,902	Same as S_3 with−sign
	P_2^6	−0.086,287		+0.381,538	+0.618,461	+0.109,364	
	P_2^7	−0.065,405		+0.267,692	+0.732,307	+0.105,790	
	P_2^8	−0.042,140		+0.163,076	+0.836,923	+0.088,294	
	P_2^9	−0.019,377		+0.072,307	+0.927,692	+0.053,992	
THIRD SPAN.	P_3^1	+0.013,717		−0.050,480		+0.924,384	+0.075,616
	P_3^2	+0.021,948		−0.086,769		+0.829,014	+0.170,986
	P_3^3	+0.025,475		−0.093,749		+0.716,712	+0.283,288
	P_3^4	+0.025,083	Same as S_1 with−sign	−0.092,307	Same as S_2 with+sign	+0.590,301	+0.409,699
	P_3^5	+0.021,556		−0.079,327		+0.452,603	+0.547,397
	P_3^6	+0.015,676		−0.057,691		+0.306,438	+0.693,562
	P_3^7	+0.008,230		−0.030,288		+0.154,630	+0.845,370

The computation of the above table is very simple and easy. Thus:—

$S_1 = \dfrac{M_2}{l_1} + P_1\,(1 - k_1)$, and numerically becomes, for loads in the first span,

$$S_1 = \left(\frac{-7.111+}{240} + \frac{210}{240}\right) P_1^1 = +0.845\ P_1^1.$$

$$S_1 = \left(\frac{-13.545}{240} + \frac{180}{240}\right) P_1^2 = +0.693\ P_1^2, \text{ etc., etc.}$$

For loads in the second and third spans, we have merely $\dfrac{M_2}{l_1}$, or the moment over the second support divided by the length of the first span.

$S_2 = \dfrac{M_3 - M_2}{l_2} + P_2\,(1 - k_2)$, for loads in the second span, and

$S_2 = \dfrac{M_3 - M_2}{l_2}$, for loads in the other spans.

$S_3 = \dfrac{-M_2}{l_3} + P_3\,(1 - k_3)$, for loads in the third span, and

$S_3 = \dfrac{-M_2}{l_3}$, for loads in the other spans.

It is necessary to compute only S_1, S_2 and S_3 to fill out the table of shears, since $S_1 + S'_2 = 0$ or P_1, $S_2 + S'_3 = 0$ or P_2, etc., but it is better to compute S'_1, S'_2 and S'_3, as a check.

TABLE c—VALUES OF M_x.

FIRST SPAN.

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
		$\overset{1}{M}{}^{1}_{x}$	$\overset{1}{M}{}^{2}_{x}$	$\overset{1}{M}{}^{3}_{x}$	$\overset{1}{M}{}^{4}_{x}$	$\overset{1}{M}{}^{5}_{x}$	$\overset{1}{M}{}^{6}_{x}$	$\overset{1}{M}{}^{7}_{x}$	
1	P_1^1	+ 25.361	+ 20.722	+ 16.083	+ 11.444	+ 6.805	+ 2.166	− 2.472	P_3^7
2	P_1^2	+ 20.806	+ 41.613	+ 32.420	+ 23.237	+ 14.044	+ 4.851	− 4.341	P_3^6
3	P_1^3	+ 16.421	+ 32.843	+ 49.265	+ 35.687	+ 22.109	+ 8.531	− 5.046	P_3^5
4	P_1^4	+ 12.290	+ 24.581	+ 36.872	+ 49.163	+ 31.554	+ 13.745	− 3.963	P_3^4
5	P_1^5	+ 8.498	+ 16 997	+ 25.495	+ 33.994	+ 42.493	+ 20.991	− 0 509	P_3^3
6	P_1^6	+ 5.129	+ 10.259	+ 15.388	+ 20.518	+ 25,647	+ 30.777	+ 5.907	P_3^2
7	P_1^7	+ 2.268	+ 4,536	+ 6.805	+ 9.073	+ 11.342	+ 13.610	+ 15.879	P_3^1
8	P_2^1	− 1.619	− 3.239	− 4.859	− 6.479	− 8.098	− 9.718	− 11.338	P_2^9
9	P_2^2	− 2.648	− 5.297	− 7.946	− 10.595	− 13.244	− 15 892	− 18.541	P_2^8
10	P_2^3	− 3.173	− 6.347	− 9.521	− 12.694	− 15.868	− 19.042	− 22.215	P_2^7
11	P_2^4	− 3.280	− 6.561	− 9.842	− 13.123	− 16.404	− 19.685	− 22.966	P_2^6
12	P_2^5	− 3.057	− 6.114	− 9.171	− 12 228	− 15.285	− 18.342	− 21.399	P_2^5
13	P_2^6	− 2.588	− 5.177	− 7.765	− 10.354	− 12.943	− 15.531	− 18.120	P_2^4
14	P_2^7	− 1.962	− 3.924	− 5.886	− 7.848	− 9.810	− 11.772	− 13.735	P_2^3
15	P_2^8	− 1.264	− 2.528	− 3.792	− 5.056	− 9.321	− 7.585	− 8.849	P_2^2
16	P_2^9	− 0.581	− 1.162	− 1.743	− 2.325	− 2.906	− 3.487	− 4.0 9	P_2^1
17	P_3^1	+ 0.411	+ 0.823	+ 1.234	+ 1.646	+ 2.057	+ 2.469	+ 2.880	P_1^7
18	P_3^2	+ 0.658	+ 1.316	+ 1.975	+ 2.633	+ 3.292	+ 3.950	+ 4.609	P_1^6
19	P_3^3	+ 0.764	+ 1.528	+ 2.292	+ 3.057	+ 3.821	+ 4.585	+ 5.349	P_1^5
20	P_3^4	+ 0.752	+ 1.504	+ 2.257	+ 3.009	+ 3.762	+ 4.514	+ 5.267	P_1^4
21	P_3^5	+ 0.646	+ 1.293	+ 1.940	+ 2.586	+ 3.233	+ 3.880	+ 4.527	P_1^3
22	P_3^6	+ 0.470	+ 0.940	+ 1.410	+ 1.881	+ 2.351	+ 2.821	+ 3.291	P_1^2
23	P_3^7	+ 0.246	+ 0.493	+ 0.740	+ 0.987	+ 1.234	+ 1.481	+ 1.728	P_1^1
		$\overset{3}{M}{}^{7}_{x}$	$\overset{3}{M}{}^{6}_{x}$	$\overset{3}{M}{}^{5}_{x}$	$\overset{3}{M}{}^{4}_{x}$	$\overset{3}{M}{}^{3}_{x}$	$\overset{3}{M}{}^{2}_{x}$	$\overset{3}{M}{}^{1}_{x}$	

THIRD SPAN.

TABLE *c*—Continued.

SECOND SPAN.

	1	2	3	4	5	6	7	8	9	10	11
		$\overset{2}{M}{}_x^{1}$	$\overset{2}{M}{}_x^{2}$	$\overset{2}{M}{}_x^{3}$	$\overset{2}{M}{}_x^{4}$	$\overset{2}{M}{}_x^{5}$	$\overset{2}{M}{}_x^{6}$	$\overset{2}{M}{}_x^{7}$	$\overset{2}{M}{}_x^{8}$	$\overset{2}{M}{}_x^{9}$	
1	P_1^1	− 6.202	− 5.293	− 4.385	− 3.476	− 2.567	− 1.659	− 0.750	+ 0.157	+ 1.066	P_3^7
2	P_1^2	− 11.824	− 10 103	− 8.382	− 6.662	− 4.941	− 3.220	− 1.500	+ 0.220	+ 1.941	P_3^6
3	P_1^3	− 16.244	− 13.864	− 11.485	− 9.105	− 6.725	− 4.345	− 1.965	+ 0.413	+ 2.793	P_3^5
4	P_1^4	− 18.903	− 16.133	− 13.364	− 10.595	− 7 826	− 5.056	− 2.287	+ 0.481	+ 3.250	P_3^4
5	P_1^5	− 19.198	− 16.385	− 13.573	− 10.760	− 7.948	− 5.136	− 2.323	+ 0.498	+ 3.301	P_3^3
6	P_1^6	− 16.540	− 14.117	− 11.693	− 9.270	− 6.847	− 4.424	− 2.001	+ 0.421	+ 2.844	P_3^2
7	P_1^7	− 10.337	− 8.823	− 7.308	− 5.794	− 4.279	− 2.765	− 1.251	+ 0.263	+ 1.777	P_3^1
8	P_2^1	+ 14.872	+ 12.703	+ 10.534	+ 8.364	+ 6.195	+ 4.026	+ 1.857	− 0.312	− 2.481	P_2^9
9	P_2^2	+ 3.917	+ 29.024	+ 24.132	+ 19.240	+ 14.347	+ 9.455	+ 4.563	− 0.329	− 5.221	P_2^8
10	P_2^3	− 3.420	+ 18.548	+ 40.517	+ 32 487	+ 24.456	+ 16.425	+ 8.394	+ 0.364	− 7.666	P_2^7
11	P_2^4	− 7.693	+ 10.860	+ 29.413	+ 47 967	+ 36.521	+ 25.075	+ 13.629	+ 2.183	− 9.263	P_2^6
12	P_2^5	− 9.456	+ 5.543	+ 20.543	+ 35.543	+ 50.543	+ 35.543	+ 20 513	+ 5.543	− 9.456	P_2^5
		$\overset{2}{M}{}_x^{9}$	$\overset{2}{M}{}_x^{8}$	$\overset{2}{M}{}_x^{7}$	$\overset{2}{M}{}_x^{6}$	$\overset{2}{M}{}_x^{5}$	$\overset{2}{M}{}_x^{4}$	$\overset{2}{M}{}_x^{3}$	$\overset{2}{M}{}_x^{2}$	$\overset{2}{M}{}_x^{1}$	

SECOND SPAN.

$\overset{1}{M}_x = S_1\ x_1 - P_1\ (x_1 - a_1)$ for loads in the first span, and $\overset{1}{M}_x = S_1\ x_1$ for loads in the other spans. $\overset{3}{M}_x = S_3\ x_3 - P_3\ (x_3 - a_3)$ for loads in the third span, and $\overset{3}{M}_x = S_3\ x_3$ for loads in the other spans.

If there were no equal spans, then Table *c* would have the number of apices squared computations, making its formation tedious, if many spans were considered, but the great worth of the table, when once computed, overbalances the hard labor in its formation.

Having Table *c* before us, we can tell at once which of the apices must be loaded, to produce a certain result, in any particular chord member.

MAXIMUM MOMENTS.

Table (***c***) enables one to find the maximum bending moments for any chord piece, with comparatively little labor.

Dead load—

For the dead, or static loads, of the structure, the *maximum bending moment for any chord member is found by taking the algebraic sum of the quantities in the proper column of Table* (***c***) (*each co-efficicient is, of course, multiplied by its* ***P***).

For example, suppose the maximum bending moment at the second panel of the first span, or, better, the second panel point of the first span, is desired: Multiply each co-efficient in column 3 of Table (***c***), by its proper ***P***, and take the algebraic sum of the products.

Live load—

For the live or moving load, *the maximum negative moment at any panel point is found by taking the sum of the negative co-efficients (multiplied by their proper* ***P's***) *in the proper column of Table* (***c***), *and vice versa for the maximum positive bending moment.*

For example, suppose the maximum negative bending moment at the second panel point is desired: Take the sum of the negative products in column 3 of Table (***c***). For positive moment, take the sum of the positive products.

If the ***P's*** are equal, the work is somewhat easier, as the co-efficients can be at once summed, and then multiplied by the common value of ***P***.

The maximum bending moments over the supports are obtained in the same manner from Table (***a***).

MAXIMUM SHEAR.

The maximum shear is obtained from Table (***b***).

Dead load—

The maximum shear at any panel point is found by:

First.—*Taking the algebraic sum of the co-efficients (multiplied by their **P's**) of all the loads outside of the span in which the apex considered lies.*

Second.—*Considering the span in which the apex lies alone, and using for the left reactions the co-efficients as found in the column giving the values of **S**.*

For example, take the second apex of the first span. The maximum shear is found: First, by taking the algebraic sum of the co-efficients in column S_1 for the second and third spans. Second, by taking the sum of the co-efficients opposite P_1^2, P_1^3 to P_1^7, inclusive, and *decreasing* it by the co-efficient opposite P_1^1 in column S_2'.

Live load—

For *positive* or *negative* shear:

First.—*Take the sum of the positive or negative co-efficients (multiplied by their **P's**) for spans in which the apex does not lie.*

Second.—*Consider the span in which the apex lies alone, and use for left reactions the positive or negative co-efficients (multiplied by their **P's**) as given in the proper column of Table (**b**).*

For example, find the maximum positive shear at the second apex of the first span:

First.—Take sum of co-efficients (multiplied by their ***P's***) in column S_1, opposite third span.

Second.—Take sum of co efficients (multiplied by their ***P's***), opposite P_1^2 to P_1^7, inclusive.

Knowing the maximum shears and bending moments, the maximum stresses are then easily obtained.

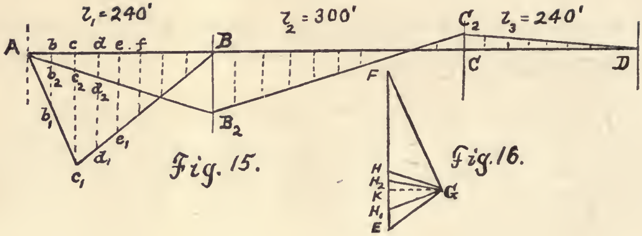


Fig. 15.

Fig. 16.

The following is a graphical solution of the same example. Tables (b) and (c) can be filled by means of graphics, as can also Table (a), by using *Prof. Greene's Method of Area Moments*, which is fully explained in *Part II* of *Greene's Trusses and Arches.*

Let it be supposed that Table (a) has been filled, either by computation or by *Greene's Method of Area Moments;* we have, then, the bending moments over the supports, or the "pier ordinates." Consider only a single concentration at the second apex of the first span, and call it P_1^2; then, if the first span were discontinuous, the moment polygon $A\ c_1\ B$ would enable us to determine the bending moments at other apices of the span; but the girder is not discontinuous, and the load P_1^2 induces a negative moment over the second support, which may be designated by $B\ B_2$, then the other negative moments are given by the ordinates in the triangle or polygon $A\ B\ B_2$. The difference of the ordinates of the two polygons determines the magnitude and kind of bending moment at each apex of the first span. The load P_1^2 induces a positive moment over the second support, which may be designated by $C\ C_2$, then the ordinates between the lines $B\ C$ and $B_2\ C_2$ will give us the moments at the apices of the second span. The

moments in the third span are given by the ordinates between the lines $C_2 D$ and $C D$.

Having given an outline of the method, we will now proceed to give it in detail.

First.—Assume some value for P_1^2, as unity, ten or one hundred.

Second.—Form the stress diagram, Fig. 16, assuming for the pole distance some value as unity, ten or one hundred. Assume the pole in such a position, that the closing line $A B$ to the equilibrium polygon $A c_1 B$ shall be horizontal, and construct the equilibrium polygon $A c_1 B$.

Third.—From Table (a), we find the bending moment over the second support to be $-13.54 P_1^2$; now, if $P_1^2=1$, and the pole distance $=1$, then the ordinate $B B_2=13.54$, laid off to the scale of $A B$. Draw $A B_2$. Scale the ordinates $b_1 b_2$, $c_1 c_2$, etc., and place the results in Table (c), column 3, opposite P_1^1, P_2^2, etc. The results will be found to agree with those computed.

Fourth.—From Table (a), we find that $C C_2=+3.762 P_1^2$; hence, lay off $C C_2=3.76$ to scale of $A B$, and draw $B_2 C_2$ and $C_2 D$ and fill out the remainder of column 3, Table (c), by scaling the ordinates between these lines and the horizontal.

In like manner, each column of Table (c) can be filled in a comparatively short time.

To fill Table (b), proceed as follows:

First.—Draw $G H$, Fig. 16, parallel to $A B_2$, then $E H=S_1$ and $F H=S_2'$.

Second.—Draw $G H_1$ parallel to $B_2 C_2$, then $K H_1=S_2 = -S_3'$.

Third.—Draw $G H_2$ parallel to $C_2 D$, then $K H_2=-S_3 = +S_4'$.

Proceed in like manner with each load.

* EXAMPLE 2—VARIABLE I.

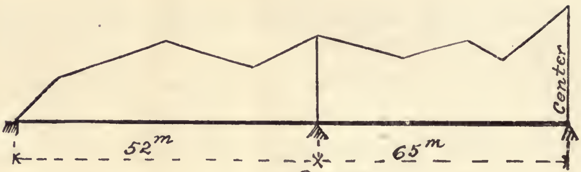


Fig. 19.

$l_1=52$, $l_2=65$, $l_3=65$, $l_4=52$. $s=4$. $r=1, 2, 3$ and 4.
$m=1, 2, 3, 4$ and 5.

$$\overset{1}{I}_0=240=\overset{4}{I}_l \qquad \overset{1}{I}_5=300=\overset{4}{I}_3$$
$$\overset{1}{I}_1=280=\overset{4}{I}_7 \qquad \overset{1}{I}_6=360=\overset{4}{I}_2$$
$$\overset{1}{I}_2=340=\overset{4}{I}_6 \qquad \overset{1}{I}_7=420=\overset{4}{I}_1$$
$$\overset{1}{I}_3=280=\overset{4}{I}_5 \qquad \overset{1}{I}_l=520=\overset{4}{I}_0$$
$$\overset{1}{I}_4=240=\overset{4}{I}_4$$

$$\overset{r}{\triangle}_v=\frac{\overset{r}{I}_l}{\overset{r}{I}_{v-1}}-\frac{\overset{r}{I}_l}{\overset{r}{I}_v}$$

$$w_1=6.7$$
$$w_2=6.7$$
$$w_3=2.2$$
$$w_4=6.7$$

$$\overset{2}{I}_0=520=\overset{3}{I}_l \qquad \overset{2}{I}_5=300=\overset{2}{I}_8 \qquad \overset{2}{I}_{10}=360=\overset{3}{I}_3$$
$$\overset{2}{I}_1=420=\overset{3}{I}_{12} \qquad \overset{2}{I}_6=360=\overset{3}{I}_7 \qquad \overset{2}{I}_{11}=420=\overset{3}{I}_2$$
$$\overset{2}{I}_2=360=\overset{3}{I}_{11} \qquad \overset{2}{I}_7=300=\overset{3}{I}_6 \qquad \overset{2}{I}_{12}=480=\overset{3}{I}_1$$
$$\overset{2}{I}_3=300=\overset{3}{I}_{10} \qquad \overset{2}{I}_8=240=\overset{3}{I}_5 \qquad \overset{2}{I}_l=560=\overset{3}{I}_0$$
$$\overset{2}{I}_4=240=\overset{3}{I}_9 \qquad \overset{2}{I}_9=300=\overset{3}{I}_4$$

Determine the value of $\boldsymbol{M_2}$.

From $(\boldsymbol{A})$,

$$M_2=\frac{-1}{(d_5\ \beta_1'')}\left\{\begin{array}{l}(A_2+X_2'+X_1'')\ d_4\\(A_3+X_3'+X_2''+B_2)\ d_3\\(A_4+X_4'+X_3''+B_3)\end{array}\right\}+\frac{-d_4\,\beta_3''\,\beta_2''}{(c_5\,\beta_4)\,\beta_3\,\beta_2}\ B_1$$

In which the several terms have the following values:

---

* See page 131, *Allgemeine Theorie und Berechnung der continuirlichen und einfachen Träger*, by J. Jacob Weyrauch, Ph. D.

First Span.

$l_1 = 52. \qquad e_l = e_8 \qquad I_l = I_8.$

e_v		$\triangle_v$		$\triangle_v \dfrac{e_v^3}{l_1}$	
$e_1 =$	10.50	$\triangle_1 =$	+ 0.300	+	6
$e_2 =$	13.75	$\triangle_2 =$	+ 0.337	+	17
$e_3 =$	31.25	$\triangle_3 =$	− 0.337	−	198
$e_4 =$	34.50	$\triangle_4 =$	− 0.300	−	237
$e_5 =$	43.50	$\triangle_5 =$	+ 0.433	+	685
$e_6 =$	46.00	$\triangle_6 =$	+ 0.289	+	541
$e_7 =$	48.00	$\triangle_7 =$	+ 0.201	+	427
$e_8 =$	50.00	$\triangle_8 =$	+ 0.243	+	584
				+	1825

$$F_1'' = + \frac{1825}{52} = + 35.0961$$

Second Span.

$l_2 = 65. \qquad e_l = e_{13}. \qquad I_l = I_{13}.$

e_v		$\triangle_v$		$\triangle_v \dfrac{l^3-(l-e_v)^3}{l}$		$\triangle_v e_v^2 \left(3-2\dfrac{e_v}{l}\right)$		$\triangle_v \dfrac{e_v^3}{l}$	
$e_1 =$	1.50	$\triangle_1 =$	− 0.256	−	73	−	2	−	0
$e_2 =$	3.00	$\triangle_2 =$	− 0.222	−	124	−	6	−	0
$e_3 =$	4.75	$\triangle_3 =$	− 0.311	−	267	−	20	−	1
$e_4 =$	6.75	$\triangle_4 =$	− 0.467	−	553	−	59	−	2
$e_5 =$	19.25	$\triangle_5 =$	+ 0.467	+	1285	+	417	+	51
$e_6 =$	23.50	$\triangle_6 =$	+ 0.311	+	972	+	391	+	62
$e_7 =$	42.75	$\triangle_7 =$	− 0.311	−	1261	−	957	−	374
$e_8 =$	47.00	$\triangle_8 =$	− 0.467	−	1931	−	1603	−	746
$e_9 =$	56.25	$\triangle_9 =$	+ 0.467	+	1968	+	1876	+	1278
$e_{10} =$	58.75	$\triangle_{10} =$	+ 0.311	+	1313	+	1279	+	970
$e_{11} =$	60.75	$\triangle_{11} =$	+ 0.222	+	938	+	927	+	765
$e_{12} =$	62.25	$\triangle_{12} =$	+ 0.167	+	706	+	701	+	620
$e_{13} =$	63.50	$\triangle_{13} =$	+ 0.166	+	701	+	700	+	657
				+	3674	+	3644	+	3280

$$\frac{+3674}{65} = +56.524 = F_2. \qquad \frac{+3644}{65} = +56.061 = F_2'.$$

$$\frac{+3280}{65} = +50.461 = F_2''.$$

Third Span.

$l_3 = 65. \qquad e_l = e_{13}. \qquad I_l = I_{13}.$

e_v	$\triangle_v$	$\triangle_v \dfrac{l^3-(l-e_v)^3}{l}$	$\triangle_v e_v^2 \left(3-2\dfrac{e_v}{l}\right)$	$\triangle_v \dfrac{e_v^3}{l}$
$e_1 =$ 1.50	$\triangle_1 = -$ 0.154	$-$ 44	$-$ 1	$-$ 0
$e_2 =$ 2.75	$\triangle_2 = -$ 0.155	$-$ 80	$-$ 3	$-$ 0
$e_3 =$ 4.25	$\triangle_3 = -$ 0.206	$-$ 160	$-$ 11	$-$ 0
$e_4 =$ 6.25	$\triangle_4 = -$ 0.289	$-$ 319	$-$ 32	$-$ 1
$e_5 =$ 8.75	$\triangle_5 = -$ 0.433	$-$ 644	$-$ 90	$-$ 4
$e_6 =$ 18.00	$\triangle_6 = +$ 0.433	$+$ 1138	$+$ 343	$+$ 39
$e_7 =$ 22.25	$\triangle_7 = +$ 0.289	$+$ 874	$+$ 331	$+$ 49
$e_8 =$ 41.50	$\triangle_8 = -$ 0.289	$-$ 1163	$-$ 858	$-$ 318
$e_9 =$ 45.75	$\triangle_9 = -$ 0.433	$-$ 1782	$-$ 1443	$-$ 638
$e_{10} =$ 58.25	$\triangle_{10} = +$ 0.433	$+$ 1827	$+$ 1775	$+$ 1316
$e_{11} =$ 60.25	$\triangle_{11} = +$ 0.289	$+$ 1220	$+$ 1202	$+$ 972
$e_{12} =$ 62.00	$\triangle_{12} = +$ 0.206	$+$ 870	$+$ 865	$+$ 755
$e_{13} =$ 63.50	$\triangle_{13} = +$ 0.238	$+$ 1005	$+$ 1004	$+$ 937
		$+$ 2742	$+$ 3082	$+$ 3107

$$\frac{+2742}{65} = +42.215 = F_3. \qquad \frac{+3082}{65} = +47.415 = F_3'.$$

$$\frac{+3107}{65} = +47.800 = F_3''.$$

Fourth Span.

$l_4 = 52. \qquad e_l = e_8. \qquad I_l = I_8$

e_v	$\triangle_v$	$\triangle_v \dfrac{l^3-(l-e_v)^3}{l}$	$\triangle_v e_v^2 \left(3-2\dfrac{e_v}{l}\right)$
$e_1 =$ 2.00	$\triangle_1 =$ 0.110	$-$ 33	$-$ 1
$e_2 =$ 4.00	$\triangle_2 =$ 0.095	$-$ 55	$-$ 4
$e_3 =$ 6 00	$\triangle_3 =$ 0 134	$-$ 111	$-$ 13
$e_4 =$ 8.50	$\triangle_4 =$ 0.200	$-$ 224	$-$ 39
$e_5 =$ 17.50	$\triangle_5 =$ 0.143	$+$ 274	$+$ 102
$e_6 =$ 20.75	$\triangle_6 =$ 0.151	$+$ 320	$+$ 143
$e_7 =$ 38.25	$\triangle_7 =$ 0.151	$-$ 401	$-$ 337
$e_8 =$ 41.50	$\triangle_8 =$ 0.143	$-$ 384	$-$ 346
		$-$ 614	$-$ 495

$$\frac{-614}{52} = -11.807 = F_4 \qquad \frac{-495}{52} = -9.519 = F_4'.$$

Collecting the values of $\boldsymbol{F}$, we have

$$
\begin{array}{lll}
 & & F_1'' = +\ 35.096 \\
F_2 = +\ 56.523 & F_2' = +\ 56.061 & F_2'' = +\ 50.461 \\
F_3 = +\ 42.215 & F_3' = +\ 47.415 & F_3'' = +\ 47.800 \\
F_4 = -\ 11.807 & F_4' = -\ \ 9.519 &
\end{array}
$$

From $(\boldsymbol{e})$,

$$\theta_1 = 6\ E\ \overset{1}{I_l} = 520\ E' \text{ (by making } E'=6\ E)$$

$$\theta_2 = 6\ E\ \overset{2}{I_l} = 560\ E' \text{ (by making } E'=6\ E)$$

$$\theta_3 = 6\ E\ \overset{3}{I_l} = 520\ E' \text{ (by making } E'=6\ E)$$

$$\theta_4 = 6\ E\ \overset{4}{I} = 240\ E' \text{ (by making } E'=6\ E)$$

From $(\boldsymbol{m})$,

$$\beta_2 = \theta_3\ (l_2 + F_2') = 520\ (121.061)\ E' = \ldots\ldots\ 62{,}951.7\ E'$$

$$\beta_3 = \theta_4\ (l_3 + F_3') = 240\ (112.415)\ E' = \ldots\ldots\ 26{,}979.6\ E'$$

From $(\boldsymbol{n})$,

$$\beta_2' = \theta_2\ (l_1 + F_1'') + \theta_1\ (l_2 + F_2)$$
$$= 560\ (87.096)\ E' + 520\ (121.523) E' = \ldots\ 111{,}965.7\ E'$$

$$\beta_3' = \theta_3\ (l_2 + F_2'') + \theta_2\ (l_3 + F_3)$$
$$= 520\ (115.461)\ E' + 560\ (107.215)\ E' = 120{,}080.1\ E'$$

$$\beta_4' = \theta_4\ (l_3 + F_3'') + \theta_3\ (l_4 + F_4)$$
$$= 240\ (112.800)\ E' + 520\ (40.193)\ E' = 47{,}972.3\ E'$$

From $(\boldsymbol{o})$,

$$\beta_2'' = \theta_1\ (l_2 + F_2') = 520\ (121.061)\ E' = \ldots\ldots\ 62{,}951.7\ E'$$

$$\beta_3'' = \theta_2\ (l_3 + F_3') = 560\ (112.415)\ E' = \ldots\ldots\ 62{,}952.4\ E'$$

$$\beta_4'' = \theta_3\ (l_4 + F_4') = 520\ (42.481)\ E' = \ldots\ldots\ 22{,}090.1\ E'$$

Since $k_r = \dfrac{a_r}{l_r}$, $\Sigma\ P_r\ l_r\ (1 - k_r) = \Sigma\ P_r\ (l_r - a_r)$. Now, we have, from $(\boldsymbol{k})$, for a uniform load, $\Sigma\ P_r = \displaystyle\int_0^{l_r} w_r\ d\,a_r$,

$$\therefore\ \Sigma\ P_r\ l_r\ (1 - k_r) = \int_0^{l_r} w_r\ d\ a_r\ (l_r - a_r) = \frac{1}{2}\ w_r\ l_r^2, \text{ and we}$$

can write

$$\Sigma\, P_1\, l_1\, (1-k_1) = \frac{1}{2}\, w_1\, l_1^2 = \frac{1}{2}\, 6.7 \cdot 2704 = \ldots\ldots \quad 9,058.4$$

$$\Sigma\, P_2\, l_2\, (1-k_2) = \frac{1}{2}\, w_2\, l_2^2 = \frac{1}{2}\, 6.7 \cdot 4225 = \ldots\ldots \quad 14,153.7$$

$$\Sigma\, P_3\, l_3\, (1-k_3) = \frac{1}{2}\, w_3\, l_3^2 = \frac{1}{2}\, 2.2 \cdot 4225 = \ldots\ldots \quad 4,647.5$$

$$\Sigma\, P_4\, l_4\, (1-k_4) = \frac{1}{2}\, w_4\, l_4^2 = \frac{1}{2}\, 6.7 \cdot 2704 = \ldots\ldots \quad 9,058.4$$

Also,

$$\Sigma\, P_r\, (e_v - a_r)^3 = \int_0^{e_v} w_r\, d\, a_r\, (e_v - a_r)^3 = \frac{1}{4}\, w_r\, e_v^4$$

$$\Sigma\, P_r\, (e_v - a_r)^2 = \int_0^{e_v} w_r\, d\, a_r\, (e_v - a_r)^2 = \frac{1}{3}\, w_r\, e_v^3$$

Substituting these values in (f) and (g), they become

$$H_r = \sum_{v=l_r}^{v=1} \overset{r}{\underset{v}{\triangle}} \left\{ \frac{w_r\, e_v^4}{4\, l_r} + \frac{3\, (l_r - e_v)}{l_r}\, \frac{w_r\, e_v^3}{3} \right\}$$

$$= \sum_{v=l_r}^{v=1} \overset{r}{\underset{v}{\triangle}}\, \frac{w_r\, e_v^3}{4} \left\{ 4 - 3\, \frac{e_v}{l_r} \right\}$$

$$H_r' = \sum_{v=l_r}^{v=1} \overset{r}{\underset{v}{\triangle}} \left\{ \frac{w_r\, e_v^4}{4\, l_r} - \frac{3\, e_v}{l_r}\, \frac{w_r\, e_v^3}{3} \right\}$$

$$= \sum_{v=l_r}^{v=1} \overset{r}{\underset{v}{\triangle}}\, \frac{3\, w_r}{4} \left\{ - \frac{e_v^4}{l_r} \right\}$$

VALUES OF H AND H'.

	l_2	l_2		l_3		l_4
v	$\Delta_v \frac{e_v^4}{l}$	$\Delta_v e_v^3 \left(4 - 3\frac{e_v}{l}\right)$	$\Delta_v \frac{e_v^4}{l}$	$\Delta_v e_v^3 \left(4 - 3\frac{e_v}{l}\right)$	$\Delta_v \frac{e_v^4}{l}$	$\Delta_v e_v'^3 \left(4 - 3\frac{e_v'}{l}\right)$
1	+ 70	— 3	— 0	— 2	— 0	— 3
2	+ 232	— 23	— 0	— 12	— 0	— 23
3	— 6180	— 126	— 2	— 60	— 1	— 106
4	— 8173	— 530	— 15	— 262	— 7	— 431
5	+ 29815	+ 10365	+ 986	— 1043	— 39	+ 2292
6	+ 24884	+ 11767	+ 1459	+ 8003	+ 699	+ 3781
7	+ 20514	— 49249	— 15980	+ 9464	+ 1090	— 15153
8	+ 29207	— 88767	— 35059	— 43059	— 13188	— 16412
9		+ 117518	+ 71927	— 78303	— 29183	
10		+ 81258	+ 57000	+ 112239	+ 76693	
11		+ 59538	+ 46518	+ 77063	+ 58589	
12		+ 45396	+ 38580	+ 55895	+ 46830	
13		+ 45445	+ 41523	+ 65156	+ 59533	
	H_1'	H_2	H_2'	H_3	H_3'	H_4
	$-67777\, w_1$	$+58147\, w_2$	$-155203\, w_2$	$+51270\, w_3$	$-150762\, w_3$	$-6514\, w_4$

$$\therefore \quad H_2 = +\ 389{,}584.9. \qquad H_1' = -\ \ 454{,}105.9.$$

$$H_3 = +\ 112{,}794.0. \qquad H_2' = -\ 1{,}039{,}860.1.$$

$$H_4 = -\ \ 43{,}643.8. \qquad H_3' = -\ \ 331{,}676.4.$$

From ($\boldsymbol{h}$),

$$X_2 = X_2' + X_1'' \left\{ \begin{array}{l} -\ F_2'\ \Sigma\ P_2\ l_2\ (1-k_2)\ \theta_1 = \\ \qquad -56.06\,(14{,}153.7)\,520\,E' = -\,412{,}597{,}339\,E' \\ -\ 2\ F_1''\ \Sigma\ P_1\ l_1\ (1-k_1)\ \theta_2 = \\ \qquad -2{\cdot}35{,}09\,(9{,}058.4)\,560\,E' = -\,356{,}002{,}336\,E' \\ +\ H_2\,\theta_1 = +\,389{,}584.9\,(520)\,E' = +\,202{,}584{,}148\,E' \\ -H_1'\,\theta_2 = +454{,}105.9\,(560)\,E' = +\,254{,}299{,}304\,E' \end{array} \right.$$

$$\overline{\qquad -\,311{,}716{,}223\,E'}$$

$$X_3 = \\ X_3' + X_2'' \left\{ \begin{array}{ll} + F_3' \, \Sigma \, P_3 \, l_3 \, (1 - k_3) \, \theta_2 = & \\ \quad -47.41 \, (4647.5) \, 560 \, E' = & - \;\; 123,389,266 \, E' \\ - 2 \, F_2'' \, \Sigma \, P_2 \, l_2 \, (1 - k_2) \, \theta_3 = & \\ \quad -2 \, (50.46) \, (14153.7) \, 520 \, E' = & - \;\; 742,763,530 \, E' \\ + H_3 \, \theta_2 = + 112,794.0 \, (560) \, E = & + \;\;\;\; 63,164,640 \, E' \\ - H_2' \, \theta_3 = + 1,039,860.1 \, (520) \, E' = & + \;\; 540,727,252 \, E' \\ \hline & - \;\; 262,260,904 \, E' \end{array} \right.$$

$$X_4 = \\ X_4' + X_3'' \left\{ \begin{array}{ll} - F_4' \, \Sigma \, P_4 \, l_4 \, (1 - k_4) \, \theta_3 = & \\ \quad +9.52 \, (9055.4) \, 520 \, E' = & + \;\;\;\; 44,842,703 \, E' \\ - 2 \, F_3'' \, \Sigma \, P_3 \, l_3 \, (1 - k_3) \, \theta_4 = & \\ \quad -47.80 \, (4647.5) \, 240 \, E' = & - \;\; 106,632,240 \, E' \\ + H_4 \, \theta_3 = - 43,643.8 \, (520) \, E' = & - \;\;\;\; 22,694,776 \, E' \\ - H_3' \, \theta_4 = + 331,676.4 \, (240) \, E' = & + \;\;\;\; 79,602,336 \, E' \\ \hline & - \;\;\;\;\;\; 4,881,977 \, E' \end{array} \right.$$

From (75), $A_r = -\frac{1}{4} \, w_r \, l_r^3 \, \theta_{r-1}$.

From (77), $B_r = -\frac{1}{4} \, w_r \, l_r^3 \, \theta_{r+1}$.

Therefore, $A_2 = -\frac{1}{4} \, w_2 \, l_2^3 \, \theta_1 = -239,198,375 \, E'$

$$A_3 = -\frac{1}{4} \, w_3 \, l_3^3 \, \theta_2 = - \;\; 84,584,500 \; E'$$

$$A_4 = -\frac{1}{4} \, w_4 \, l_4^3 \, \theta_3 = -122,469,568 \; E'$$

$$B_1 = -\frac{1}{4} \, w_1 \, l_1^3 \, \theta_2 = -131,890,304 \; E'$$

$$B_2 = -\frac{1}{4} \, w_2 \, l_2^3 \, \theta_3 = -239,198,375 \; E'$$

$$B_3 = -\frac{1}{4} \, w_3 \, l_3^3 \, \theta_4 = - \;\; 36,250,500 \; E'$$

From (p), since $c_1=0$, and $c_2=1$,

$$c_3=\frac{-2\,\beta'_2}{\beta_2}=-3.557. \qquad c_4=\frac{-2\,c_3\,\beta'_3-\beta''_2}{\beta_3}=+29.33.$$

$$c_5=\frac{-2\,c_4\,\beta'_4-c_3\,\beta''_3}{\beta_4}=\frac{-2{,}590{,}360\ E'}{\beta_4}$$

From (r), since $d_1=0$, and $d_2=1$,

$$d_3=\frac{-\beta'_4}{\beta''_3}=-1.524. \qquad d_4=\frac{-2\,d_3\,\beta'_3-\beta_3}{\beta''_2}=+5.385.$$

$$d_5=\frac{-2\,d_4\,\beta'_2-d_3\,\beta_2}{\beta''_1}=\frac{-1{,}109{,}880\ E'}{\beta''_1}$$

Therefore,

$$M_2=\frac{-1}{-1{,}109{,}880\ E'}\left\{\begin{array}{l}\left(\begin{array}{r}-239{,}198{,}375\ E'-210{,}013{,}191\ E'\\ -101{,}703{,}032\ E'\\ \hline -311{,}716{,}223\ E'\end{array}\right)\ 5.385\\ \left(\begin{array}{r}-\ 84{,}584{,}500\ E'-\ 60{,}224{,}626\ E'-239{,}198{,}375E'\\ -202{,}036{,}278\ E'\\ \hline -262{,}260{,}904\ E'\end{array}\right)(-1.524)\\ \left(\begin{array}{r}-122{,}469{,}568\ E+\ 22{,}147{,}927\ E'-36{,}250{,}500\ E'\\ -\ 27{,}029{,}904\ E'\\ \hline -\ \ 4{,}881{,}977\ E'\end{array}\right)\end{array}\right\}$$

$$+\ \frac{-5{,}385\ (2.33)}{-2{,}590{,}360\ E'}\ (-131{,}890{,}304\ E')$$

$$M_2=\frac{1}{1{,}109{,}880}\left\{\begin{array}{r}-2{,}966{,}675{,}110\\ +\ \ 893{,}130{,}719\\ -\ \ 163{,}602{,}045\end{array}\right\}+\frac{12.565}{2{,}590{,}360}(-131{,}890{,}304)$$

$$M_2=-2015.66-639.68=-2656.$$

The moments, produced by the loads in the respective spans, can be easily deduced, as follows:

First span loaded—

$$M_2=\frac{-d_4\,\beta''_3\ \beta''_2}{(c_5\ \beta_4)\ \beta_3\ \beta_2}\ B_1+\frac{-1}{(d_5\ \beta''_1)}\ X''_1\ d_4.$$

Or,

$$M_2=\left\{\begin{array}{l}+\ \dfrac{1}{1{,}109{,}880}\ (-101{,}703{,}032)\ (5{,}385)\\ \\ +\ \dfrac{12{,}565}{2{,}590{,}360}\ (-131{,}890{,}304)\end{array}\right\}=\begin{array}{r}-\ 494\\ -\ 640\\ \hline -1134\end{array}$$

Second span loaded—

$$M_2 = \frac{-1}{(d_5\ \beta_1'')} \left\{ \begin{array}{l} (A_2 + X_2')\ d_4 \\ (X_2'' + B_2)\ d_3 \end{array} \right\}$$

$$M_2 = \frac{1}{1{,}109{,}880} \left\{ \begin{array}{l} (-239{,}198{,}375 - 210{,}013{,}191)\ (5{,}385) \\ (-202{,}036{,}278 - 239{,}198{,}375)\ (-1{,}524) \end{array} \right\} = \ldots\ldots \quad -1574$$

Third span loaded—

$$M_2 = \frac{-1}{(d_5\ \beta_1'')} \left\{ \begin{array}{l} (A_3 + X_3')\ d_3 \\ (X_3'' + B_3) \end{array} \right\}$$

$$M_2 = \frac{1}{1{,}109{,}880} \left\{ \begin{array}{l} (-84{,}584{,}500 - 60{,}224{,}626)\ (-1{,}524) \\ (-27{,}029{,}904 - 36{,}250{,}500) \end{array} \right\} = \ldots\ldots \quad +142$$

Fourth span loaded—

$$M_2 = \frac{-1}{(d_4\ \beta_1'')} \left\{ A_4 + X_4' \right\}$$

$$M_2 = \frac{1}{1{,}109{,}880} \left\{ \begin{array}{l} -122{,}469{,}568 \\ +\ 22{,}147{,}927 \end{array} \right\} = \ldots\ldots \quad -90$$

Sum $= -2656$

THE SAME EXAMPLE WITH I CONSTANT.

$$M_2 = \frac{-c_2}{d_5\ l_1} \left\{ \begin{array}{l} A_2\ d_4 + B_2\ d_3 \\ A_3\ d_3 + B_3 \\ A_4 \end{array} \right\} + \frac{-d_4}{c_5\ l_4}\ B_1$$

$$c_1 = 0,\ c_2 = 1,\ c_3 = -3.6,\ c_4 = +13.4,\ c_5 = -\frac{2901.6}{52}$$

$$d_1 = 0,\ d_2 = 1,\ d_3 = -3.6,\ d_4 = +13.4,\ d_5 = -\frac{2901.6}{52}$$

$$A_2 = -\frac{1}{4}\ w_2\ l_2^3 \qquad A_3 = -\frac{1}{4}\ v_3\ l_3^3 \qquad A_4 = -\frac{1}{4}\ w_4\ l_4^3$$

$$B_1 = -\frac{1}{4}\ w_1\ l_1^3 \qquad B_2 = -\frac{1}{4}\ w_2\ l_2^3 \qquad B_3 = -\frac{1}{4}\ v_3\ l_3^3$$

Then,

$$M_2 = \frac{1}{2901.6} \left\{ \begin{array}{ll} -\frac{1}{4}\ w_3\ l_2^3\ (\ 13.4) & -\frac{1}{4}\ w_2\ l_2^3\ (-3.6) \\ -\frac{1}{4}\ v_3\ l_3^3\ (-3.6) & -\frac{1}{4}\ v_3\ l_3^3 \\ -\frac{1}{4}\ w_4\ l_4^3 & -\frac{1}{4}\ w_1\ l_1^3\ (\ 13.4) \end{array} \right\}$$

$$M_2 = \frac{1}{2901.6} \cdot \left\{ \begin{array}{l} -919{,}993.7\,w + 247{,}162.5\,w \\ +247{,}162.5\,v - 68{,}656.2\,v \\ -35{,}152{,}0\,w - 471{,}036.8\,w \end{array} \right\} = -2587$$

The partial moments are easily obtained, as follows:

First span loaded—

$$M_2 = \frac{1}{2901.6}\,(-471{,}036.8\,w) = \ldots\ldots\ldots -1088$$

Second span loaded—

$$M_2 = \frac{1}{2901.6}\,(-919{,}993.7\,w + 247{,}162.5\,w) = \ldots -1554$$

Third span loaded—

$$M_2 = \frac{1}{2901.6}\,(247{,}162.5\,v - 68{,}656.2\,v) = \ldots\ldots +136$$

Fourth span loaded—

$$M_2 = \frac{1}{2901.6}\,(-35{,}126.0\,w) = \ldots\ldots\ldots -81$$

$$\text{Sum} = -2587$$

Spans loaded.	*Variable **I**.*	*Constant **I**.*	*Difference.*	*Per cent. of Variable **I** Results.*
FIRST.	— 1134	— 1088	— 46	.041
SECOND.	— 1574	— 1554	— 20	.012
THIRD.	+ 142	+ 136	+ 6	.043
FOURTH.	— 90	— 81	— 9	.100
ALL.	— 2656	— 2587	— 69	.026

EXAMPLE 3.—THE SABULA DRAW.

We now propose to compare the bending moments in the Sabula Draw, considering the moment of inertia as constant and variable, respectively.

Figs. 17 & 20

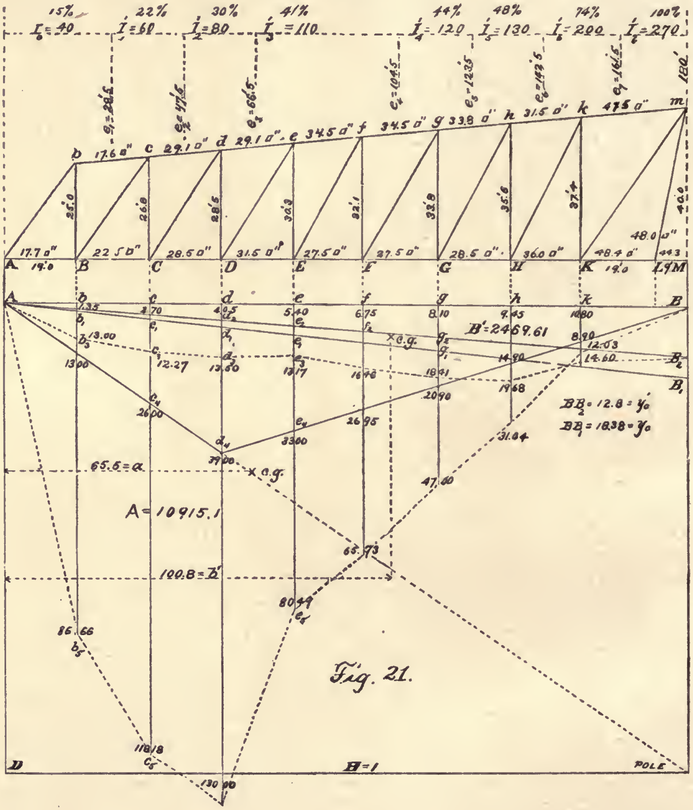


Fig. 21.

Variable moment of inertia—

From Fig. 20, is obtained the cross section of each mem-

ber, and the length of each vertical; the moments of inertia for each section are readily deduced as follows: neglecting the moment of inertia of the section of the member about its own axis, it being very small, in comparison with the moment of inertia of the truss section.

FIRST SPAN. SECOND SPAN.

$$\overset{1}{I}_0=(17.7+17.6)\ (12.5\ )^2\div 144=\ 40=\ \overset{2}{I}_1$$
$$\overset{1}{I}_1=(29.1+22.5)\ (13.4\ )^2\div 144=\ 60=\ \overset{2}{I}_6$$
$$\overset{1}{I}_2=(29.1+28.5)\ (14.25)^2\div 144=\ 80=\ \overset{2}{I}_5$$
$$\overset{1}{I}_3=(34.5+31.5)\ (15.15)^2\div 144=110=\ \overset{2}{I}_4$$
$$\overset{1}{I}_3=(34.5+27.5)\ (16.05)^2\div 144=110=\ \overset{3}{I}_4$$
$$\overset{1}{I}_4=(33.8+27.5)\ (16.9\ )^2\div 144=120=\ \overset{2}{I}_3$$
$$\overset{1}{I}_5=(31.5+28.5)\ (17.8\ )^2\div 144=130=\ \overset{2}{I}_2$$
$$\overset{1}{I}_6=(47.5+36.0)\ (18.7\ )^2\div 144=200=\ \overset{2}{I}_1$$
$$\overset{1}{I}_l=(47.5+48.0)\ (20.0\ )^2\div 144+270=\ \overset{2}{I}_0$$

FIRST SPAN.		SECOND SPAN.	
$e_1=28.5$	$e_4=104.5$	$e_1=18.5$	$e_4=75.5$
$e_2=47.5$	$e_5=123.5$	$e_2=37.5$	$e_5=113.5$
$e_3=66.5$	$e_6=142.5$	$e_3=56.5$	$e_6=132.5$
$e_7=161.5$		$e_7=151.5$	

From (124), we have,

$$M_2=\frac{1}{2\ \beta_2'}\ (A_2+B_1+X_2'+X_1'')$$

In which,

$$X_2'=H_2\ \theta_1-F_2'\ \Sigma\ P_2\ l_2\ (1-k_2)\ \theta_1$$

$$X_1''=-\ H_1'\ \theta_2-2\ F_1''\ \Sigma\ P_1\ l_1\ (1-k_1)\ \theta_2$$

$$H_2=\sum_{v=l_2}^{v=1}\ \overset{2}{\triangle}_v\left\{\frac{\Sigma\ P_2\ (e_v-a_2)^3}{l_2}+\frac{3\ (l_2-e\ )}{l_2}\cdot\Sigma\ P_2\ (e_v-a_2)^2\right\}$$

$$H_1' = \sum_{v=l_1}^{v=1} \overset{1}{\triangle}_v \left\{ \frac{\Sigma P_1 (e_v - a_1)^3}{l_1} - \frac{3 e_v}{l_1} \Sigma P_1 (e_v - a_1)^2 \right\}$$

$$\beta_2' = 6 E \overset{2}{I}_t (l_1 + F_1'') + 6 E \overset{1}{I}_t (l_2 + F_2)$$

$$F_1'' = \sum_{v=l_1}^{v=1} \overset{1}{\triangle}_v \frac{e_v^3}{l_1^2}$$

$$F_2' = \sum_{v=l_2}^{v=1} \overset{2}{\triangle}_v e_v^2 \left\{ \frac{3}{l_2} - \frac{2 e_v}{l_2^2} \right\}$$

$$F_2 = \sum_{v=l_2}^{v=1} \overset{2}{\triangle}_v \left\{ \frac{l_2^3 - (l_2 - e_v)^3}{l_2^2} \right\}$$

$$\overset{2}{\triangle}_v = \frac{\overset{2}{I}_t}{\overset{2}{I}_{v-1}} - \frac{\overset{2}{I}_t}{\overset{2}{I}_v} \qquad \overset{1}{\triangle}_v = \frac{\overset{1}{I}_t}{\overset{1}{I}_{v-1}} - \frac{\overset{1}{I}_t}{\overset{1}{I}_v}$$

We will first compute those co-efficients that do not depend upon the loading. For a uniform load over all,

$$H_2 = \sum_{v=l_2}^{v=1} \overset{2}{\triangle}_v \frac{w_2 e_v^3}{4} \left\{ 4 - 3 \frac{e_v}{l_2} \right\}$$

$$H_1 = \sum_{v=l_1}^{v=1} \overset{1}{\triangle}_v \frac{3 w_1}{4} \left\{ - \frac{e_v^4}{l_1} \right\}$$

First span—

		$-\overset{1}{\underset{r}{\triangle}} e_r^4$
$e_1 = 28.5$	$\triangle_1 = + 2.250$	$-$ 1,484,437
$e_2 = 47.5$	$\triangle_2 = + 1.125$	$-$ 5,726,996
$e_3 = 66.5$	$\triangle_3 = + 0.921$	$-$ 18,011,347
$e_4 = 104.5$	$\triangle_4 = + 0.204$	$-$ 24,327,379
$e_5 = 123.5$	$\triangle_5 = + 0.173$	$-$ 40,245,185
$e_6 = 142.5$	$\triangle_6 = + 0.727$	$-$ 299,773,934
$e_7 = 161.5$	$\triangle_7 = + 0.350$	$-$ 238,099,317
		$-$ 627,668,595

Multiplying this sum by $\frac{3\,w_1}{4\,l_1}$, we obtain $-\,2{,}615{,}286\,w_1 = H_1'$.

Second span—

		$\overset{2}{\triangle}\, e_v^3 \frac{(4\,l_2 - 3\,e_v)}{l_2}$
$e_1 = 18.5$	$\triangle_1 = -\,0.052$	$-$ 1,215
$e_2 = 37.5$	$\triangle_2 = -\,0.107$	$-$ 19,043
$e_3 = 56.5$	$\triangle_3 = -\,0.026$	$-$ 14,342
$e_4 = 75.5$	$\triangle_4 = -\,0.030$	$-$ 35,397
$e_5 = 113.5$	$\triangle_5 = -\,0.137$	$-$ 422,318
$e_6 = 132.5$	$\triangle_6 = -\,0.166$	$-$ 691,825
$e_7 = 151.5$	$\triangle_7 = -\,0.333$	$-$ 1,709,655
		$-$ 2,893,795

Multiplying this sum by $\frac{w_2}{4}$, we obtain $-723{,}449\,w_2 = H_2$.

First span— $l_2 = 180'$—

$e_1 = 28.5$	$\triangle_1 = + 2.250$	$\triangle_1 e_1^3 = +$ 52,085
$e_2 = 47.5$	$\triangle_2 = + 1.125$	$\triangle_2 e_2^3 = +$ 120,568
$e_3 = 66.5$	$\triangle_3 = + 0.921$	$\triangle_3 e_3^3 = +$ 270,847
$e_4 = 104.5$	$\triangle_4 = + 0.204$	$\triangle_4 e_4^3 = +$ 232,797
$e_5 = 123.5$	$\triangle_5 = + 0.173$	$\triangle_5 e_5^3 = +$ 325,871
$e_6 = 142.5$	$\triangle_6 = + 0.727$	$\triangle_6 e_6^3 = +$ 2,103,676
$e_7 = 161.5$	$\triangle_7 = + 0.350$	$\triangle_7 e_7^3 = +$ 1,474,299

$l_1^2 = 32{,}400 \;)\; 4{,}580{,}143 \;(\; 141.36 = F_1''.$

Second span—$l_2 = 180$.

			$(3\,e_v^2\,l_2 - 2\,e_v^3)\,\overset{2}{\underset{v}{\triangle}}$	$\left\{l_2^3 - (l_2 - e_v)^3\right\}\overset{2}{\underset{v}{\triangle}}$
$e_1 =$	18.5	$\triangle_1 =$ −0.052	— 8,951	— 84,225
$e_2 =$	37.5	$\triangle_2 =$ −0.107	— 69,967	— 314,404
$e_3 =$	56.5	$\triangle_3 =$ −0.026	— 35,440	— 102,657
$e_4 =$	75.5	$\triangle_4 =$ −0.030	— 66,521	— 140,725
$e_5 =$	113.5	$\triangle_5 =$ −0.137	— 552,403	— 758,695
$e_6 =$	132.5	$\triangle_6 =$ −0.166	— 801,442	— 950,321
$e_7 =$	151.5	$\triangle_7 =$ −0.333	—1,813,227	—1,936,284
			$l_2^2 = 32{,}400$) —3,347,951	$l_2^2 = 32{,}400$) —4,287,311
			$-103.33 = F_2'$	$-132.32 = F_2$

Next find the values of $\boldsymbol{X}_2$ and $\boldsymbol{X}_1'$, for a uniform load over all.

$\theta_1 = 270\ E'$, $\theta_2 = 40\ E'$, in which $E' = 6\ E$.

$$+\,H_2\,\theta_1 = -\,723{,}449\ (270)\ E'\,w_2 = \qquad -\,195{,}331{,}230\ w_2\,E'$$

$$-\,\frac{F_2'}{2}\,w_2\,l_2^2\,\theta_1 = +\,\frac{103.33}{2}\,(32{,}400)\ (270)\ E'\,w_2 = +\,451{,}965{,}420\ w_2\,E'$$

$$+\,256{,}634{,}190\ w_2\,E'$$

$$-F_1''\,\theta_2\,w_1\,l_1^2 = -141.36\,(32{,}400)(40)\,w_1\,E' = -\,183{,}202{,}560\ w_1\,E'$$

$$-H_1'\,\theta_2 = +\,2{,}615{,}286\ (40)\ w_1\,E' = \qquad +\,104{,}611{,}440\ w_1\,E'$$

$$-\ 78{,}591{,}120\ w_1\,E'$$

$$\beta_2' = 40\ (180 + 141.36)\ E' + 270\ (180 - 132.32)\ E' = +\,25{,}728\ E'.$$

$$A_2 = -\,\frac{1}{4}\,w_2\,l_2^3\ 270\ E' = -\,393{,}660{,}000\ E'\,w_2.$$

$$B_1 = -\,\frac{1}{4}\,w_1\,l_1^3\ 40\ E' = -\,58{,}320{,}000\ E'\,w_1.$$

Then,

$$M_2 = \frac{1}{51456E'}\left\{\begin{array}{l} -\,393{,}660{,}000\ w_2\ E' \\ -\ \ 58{,}320{,}000\ w_1\ E' \\ -\ \ 78{,}591{,}120\ w_1\ E' \\ +\,256{,}634{,}190\ w_2\ E' \end{array}\right\} = (\text{if } w_1 = w_2 = w) - 5322\ w.$$

First span alone loaded—

$$M_2 = \frac{1}{51456}\left\{\begin{array}{r} -\ \ 58{,}320{,}000\ w_1 \\ -\ \ 78{,}591{,}120\ w_1 \\ \hline -\ 137{,}000{,}000\ w_1 \end{array}\right\} = -\,2661\ w_1.$$

Second span alone loaded—

$$M_2 = \frac{1}{51456}\left\{\begin{array}{r} -\ 393{,}660{,}000\ w_2 \\ +\ 256{,}834{,}190\ w_2 \\ \hline -\ 137{,}000{,}000\ w_2 \end{array}\right\} = -\,2661\ w_2.$$

EXAMPLE 3*a*.

This is the same as Example 3, with the moment of inertia considered as constant.

From (142),

$$M_2 = \frac{A_2 + B_1}{2\,(l_1 + l_2)} = \frac{A_2 + B_1}{720}$$

$$A_2 = -\frac{1}{4}\ w_2\,l_2^3 = -\,145{,}800\ w_2.$$

$$B_1 = -\frac{1}{4}\ w_1\,l_1^3 = -\,145{,}800\ w_2.$$

Therefore,

$M_2 = \left\{\begin{array}{l} -\,2025\ w_1 \\ -\,2025\ w_2 \end{array}\right\}$ and if $w_1 = w_2 = w$, $M_2 = -\,4050\ w$.

No. of Loaded Span.	M_2 *Bending Moment Variable I.*	M_2 *Bending Moment Constant I.*	*Difference.*
FIRST.	$-\,2661\ w_1$	$-\,2025\ w_1$	$636\ w_1$
SECOND.	$-\,2661\ w_2$	$-\,2025\ w_2$	$636\ w_2$
BOTH.	$-\,5322\ w$	$-\,4050\ w$	$1272\ w$

EXAMPLE 4.—CONCENTRATED LOADS.

Let us consider the Sabula Draw again, but with single concentrated loads, instead of uniform loads. Supports level. $a_1=57$. $a_2=123$.

See pages 73 and 74 for the value of M_2.

From Example 3:

$\beta_2'=40\ (180+141.36)\ E'+270\ (180-132.32)\ E'=+25{,}728\ E'$.

$F_1''=+141.36$. $F_2=103.33$. $F_2=-132.32$.

$\theta_1=270\ E'$. $\theta_2=40\ E'$.

The values of H_2 and H_1' depend upon the position of the loads. Let us take a load in the first span $57'$ from the left support, and one in the second span $123'$ from the left support; then we have for H_1' and H_2, the following:

First span— $l_1=180'$. $a_1=57'$.

$$H_1'=\sum_{v=l_1}^{v=1} \triangle_v^1 \left\{\frac{\Sigma P_1\,(e_v-a_1)^2}{l_1}-\frac{3\,e_v}{l_1}\,\Sigma\,P_1\,(e_v-a_1)^2\right\}$$

$e_1=28.5$	$e-a_1$.		
$e_2=47.5$			
$e_3=66.5$	$+\ 9.5$	$\triangle_3=+0.921$	$-\ 88\ P_1$
$e_4=104.5$	$+\ 47.5$	$\triangle_4=+0.204$	$-\ 680\ P_1$
$e_5=123.5$	$+\ 66.5$	$\triangle_5=+0.173$	$-\ 1{,}292\ P_1$
$e_6=142.5$	$+\ 85.5$	$\triangle_6=+0.727$	$-\ 10{,}097\ P_1$
$e_7=161.5$	$+\ 104.5$	$\triangle_7=+0.350$	$-\ 8{,}068\ P_1$
			$-\ 20{,}225\ P_1=H_1'$.

NOTE.—e_v must always be greater than or equal to a_1.

Second span— $l_2=180'$. $a_2=123'$.

$$H_2=\sum_{v=l_2}^{v=1} \triangle_v^2 \left\{\frac{\Sigma P_2\,(e_v-a_2)^3}{l_2}+\frac{3\,(l_2-e_v)}{l_2}\,\Sigma\,P_2\,(e_v-a_2)^2\right\}$$

$$
\begin{array}{llllll}
e_1 = & 18.5 & & & & \\
e_2 = & 37.5 & e-a_2 & & & \\
e_3 = & 56.5 & & & & \\
e_4 = & 75.5 & & & & \\
e_5 = & 113.5 & & & & \\
e_6 = & 132.5 & +\ 9.5 & \triangle_6 = -0.166 & & -\ 13\ P_2 \\
e_7 = & 151.5 & +\ 28.5 & \triangle_7 = -0.333 & & -\ 172\ P_2 \\
\hline
 & & & & & -\ 185\ P_2 = H_2
\end{array}
$$

NOTE.—e_v must always be greater than or equal to a_2.

$$
X_2' = \begin{cases} + H_2\ \theta_1 = -185\ (270)\ E'\ P_2 = - & 49{,}950\ E'\ P_2 \\ - F_2'\ P_2\ (l_2-a_2)\ \theta_1 = & \\ \qquad +103.33\ (57)\ (270)\ E'\ P_2 = + & 1{,}590{,}248\ E'\ P_2 \\ \hline & +\ 1{,}540{,}298\ E'\ P_2 \end{cases}
$$

$$
X_1'' = \begin{cases} - H_1'\ \theta_2 = +20{,}225\ P_1\ (40)\ E' = + & 809{,}000\ E'\ P_1 \\ -2\ F_1''\ P_1\ (l_1-a_1)\ \theta_2 = & \\ \qquad -141.36\ (123)\ (80)\ E'\ P_1 = - & 1{,}390{,}928\ E'\ P_1 \\ \hline & -\ 581{,}928\ E'\ P_1 \end{cases}
$$

From (i) and (j),

$$A_2 = -\ P_2\ l_2^2\ (2\ k - 3\ k^2 + k^3)\ 6\ E'\ \overset{1}{I_l},$$

Or, by Table I,

$$A_2 = -\ P_2\ l_2^2\ (0.285{,}144)\ 270\ E' = -\ 2{,}494{,}269\ E'\ P_2.$$

$$B_1 = -\ P_1\ l_1^2\ (0.285{,}144)\ \ 40\ E' = -\ \ 369{,}520\ E'\ P_1.$$

Therefore,

$$
M_2 = \frac{1}{51456\ E'} \begin{Bmatrix} -\ 2{,}494{,}269\ E'\ P_2 \\ +\ 1{,}540{,}298\ E'\ P_2 \\ -\ \ 369{,}520\ E'\ P_1 \\ -\ \ 581{,}928\ E'\ P_1 \end{Bmatrix} = \begin{Bmatrix} -18.5\ P_2 \\ -18.5\ P_1 \end{Bmatrix}
$$

Or, if $P_1 = P_2$,

$$M_2 = -\ 37.0\ P.$$

First span alone loaded—

$$M_2 = \frac{1}{2\ \beta_2'}\ (B_1 + X_1'')$$

Or,

$$M_2 = \frac{1}{51456}\left\{-951{,}448\ P_1\right\} = 18.5\ P_1.$$

***Second span alone loaded*--**

$$M_2 = \frac{1}{2\,\beta_2'}\left\{A_2 + X_2'\right\}$$

Or,

$$M_2 = \frac{1}{51456}\left\{-953{,}961\ P_2\right\} = 18.5\ P_2.$$

Since $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ are symmetrical about the center of the draw, the moments over the center pier should be equal, as shown above. If our work was absolutely correct, and decimals had been used to several places, the quantities in the parentheses above would have been the same; as it is, the quotients are practically equal.

* EXAMPLE 4 BY GRAPHICS.

For any load in the first span of a two span girder, we have, from *Greene's Trusses and Arches, Part II*,

$$\Sigma\ \frac{x_1\,y_1}{E\,I_1} - \Sigma\ \frac{x_1'\,y_1'}{E\,I_1} = \Sigma\ \frac{x_2'\,y_2'}{E\,I_2}$$

In which:

$\boldsymbol{x_1}$=The distance from the left support to the ordinate $\boldsymbol{y_1}$ of the equilibrium polygon $\boldsymbol{A\ C\ B}$. Fig. 21, page 72.

$\boldsymbol{y_1}$=Any ordinate of the polygon $\boldsymbol{A\ C\ B}$.

$\boldsymbol{x_1'}$=The distance from the left support to the ordinate $\boldsymbol{y_1'}$ of the polygon $\boldsymbol{A\ B\ B_1}$.

$\boldsymbol{y_1'}$=Any ordinate of the polygon $\boldsymbol{A\ B\ B_1}$.

$\boldsymbol{x_2'}$=The distance from the *right* support to any ordinate $\boldsymbol{y_2'}$ of a polygon in the second span, similar to $\boldsymbol{A\ B\ B_1}$.

* For this excellent graphical method, the author is indebted to R. H. Brown, C. E., First Assistant Engineer of Boston Bridge Works.

y'_2=Any ordinate of the above polygon.

$\overset{1}{I}$=The moment of inertia of the cross section of the girder at any ordinate y'_1.

$\overset{2}{I}$=The moment of inertia of the cross section of the girder at any ordinate y'_2.

E=The modulus of elasticity.

Since $l_1=l_2=l$, and $\overset{1}{I}=\overset{2}{I}=I$, we have, considering E as constant,

$$\Sigma \frac{x_1\, y_1}{I} = 2\, \Sigma \frac{x'_1\, y'_1}{I}$$

Now, $\Sigma \frac{x_1 y_1}{I}$ equals an area multiplied by the distance of its center of gravity from the left support, and hence we may write $\Sigma \frac{x_1\, y_1}{I} = A\, a$, in which A represents the area and a the c. g. distance.

In like manner, $2\, \Sigma \frac{x'_1\, y'_1}{I} = 2\, B\, b$. Therefore, we may write, $A\, a = 2\, B\, b$.

With any scale, lay off the horizontal line $A\ B$, Fig. 21, and divide it into panel points at b, c, d, etc.

With any scale, preferably a large scale, lay off a load line $A\ D$, equal unity, and assume H also equal unity, then, assuming the pole in such a position that its closing line will be horizontal, construct the equilibrium polygon $A\ C\ B$ and scale its ordinates (the lengths are given in Fig. 21.)

Not knowing the proper position of the closing line, let us assume a position as $A\ B_2$, making $B\ B_2 = 12.8$, and scale the ordinates $b\ b_2$, $c\ c_2$, etc.

Divide each ordinate of the polygon $A\ C\ B$ by its proper I (given in per cent. of the center I above Fig. 21), and lay

off the results downward from $A\ B$, forming the polygon $A\ b_5\ c_5$ . . . . B. The ordinates are **86.66**, **118.18**, etc., as shown in Figure 21, page 72.

Now, find the area of this figure, either by computation or the planimeter, and also its center of gravity. The center of gravity is readily found by cutting the polygon out of stiff card board and balancing it upon a needle point.

The area $= 10915.1 = A$, and $a = 65.5$.

Proceed in like manner with the polygon $A\ B\ B_2$, deducing the figure $A\ b_3\ c_3$ . . . . B_2, which has an area of $2469.61 = B'$, and $b' = 100.8$.

Let $B\ B_1$ represent the true magnitude of the pier ordinate y_0, then from the triangles $A\ B\ B_2$ and $A\ B\ B_1$, we have,

$y_0' : y_0 :: e\ e_2 : e\ e_1$, or, $e\ e_1 = \dfrac{y_0}{y_0'}\ e\ e_2$. Then,

$$\Sigma\ \frac{x_1'\ (e\ e_1)}{I} = \frac{y_0}{y_0'}\ \Sigma\ \frac{x_2'\ (e\ e_2)}{I}, \text{ or, } B\ b = \frac{y_0}{y_0'}\ B'\ b'.$$

Hence, $A\ a = 2\ \dfrac{y_0}{y_0'}\ B'\ b'$, and $y_0 = \dfrac{(A\ a)\ y_0'}{2\ B'\ b'}$,

Which becomes

$$y_0 = \frac{10915.1 \times 65.5 \times 12.8}{2 \times 2469.61 \times 100.8} = 18.38.$$

Since $H = 1$ and $P_1^3 = 1$, the bending moment over the center pier is $-$ $\boldsymbol{18.38\ P_1^3}$. We obtained, by computation, $-$ $\boldsymbol{18.5\ P_1^3}$, which shows the graphical method to be accurate enough for all practical purposes.

It would have been more correct to have taken the ordinates y at the points where the values of I change, but the result would have been but little different.

The above computations can be very readily made by means of Thatcher's Calculating Instrument.

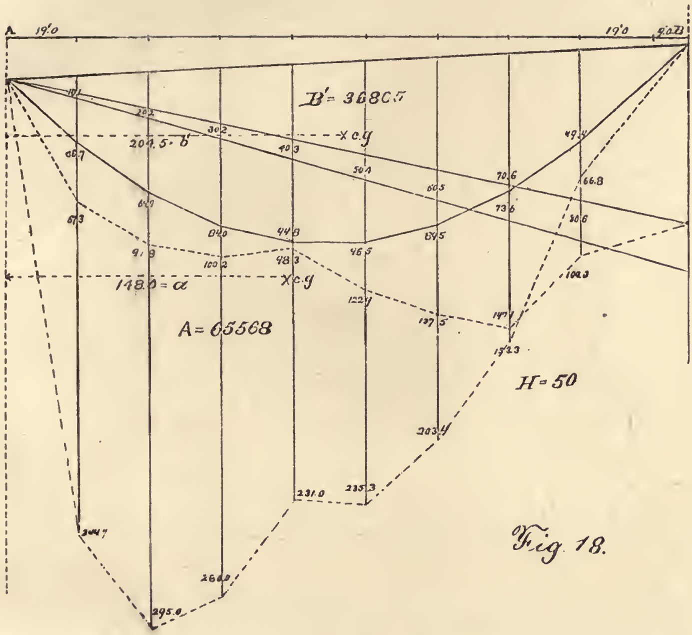


Fig. 18.

Fig. 17, page 72, and Fig. 18 are diagrams for the Sabula Draw, with a concentration at each apex in both spans.

$(P_1^1 = P_1^2 = P_2^1 = P_2^2 = \&c. = P = 23{,}000$ lbs.)

$A\ a = B\ b$, or $A\ a = \frac{y_0}{y_0'}\ B'\ b$. Therefore,

$$y_0 = \frac{(A\ a)\ y_0'}{B'\ b'} = \frac{65568 \times 148 \times 94.6}{36865 \times 204.5} = 121.800.$$

Since $H = 50$, the bending moment over the pier is $121{,}800 \times 50 =$ 6,090,000

Considering I as constant, the bending moment is 4,732,000

1,258,000 = Diff.

APPENDIX.

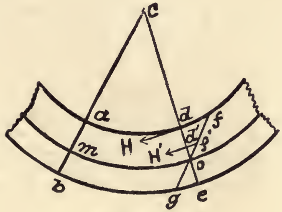


Fig. a.

* The general equation of the elastic line can be deduced as follows:

Vertical forces acting upon a girder cause a change of shape, lengthening the originally parallel fibres on one side, and shortening or compressing them on the other. Between the lengthened and shortened fibres there is a plane which undergoes no change in length; the centre line of this plane is called the *neutral axis or the elastic line.*

Thus, in Fig. (a), $m\ o$ is the neutral axis, the fibres above being compressed, and those below lengthened. Upon the three following hypotheses we shall deduce the equation of the *elastic line.*

I. All planes perpendicular to the axis before the bending or flexure, preserve, during the bending, their perpendicularity and their forms as planes.

II. The change in length of a body subjected to a force, is, within certain limits, called the elastic limits, proportional to the intensity of the force.

* See Merriman's Theory and Calculation of Continuous Beams.

III. The change of shape is so small that the length of the neutral axis is sensibly the same as its horizontal projection.

In Fig. (a), we have a longitudinal section of a portion of a bent beam; the two planes $a\ b$ and $d\ e$ origininally parallel, remaining perpendicular to the neutral axis or elastic line $m\ o$, and intersecting in c the centre of curvature. Hence, drawing $f\ g$ parallel to $a\ b$ through o, the lines $f\ d$, $g\ e$, etc., denote the elongation of the fibres, and we see, from the figure, that

$$o\ d : o\ d' :: d\ f : d'\ f' \quad \ldots \ldots \quad (22a)$$

or the change of length in the fibres is proportional to their distance from the neutral axis. *This is a consequence of the first hypothesis.*

Designating by H and H' the force acting in the fibres $d\ f$ and $d'\ f'$, *the second hypothesis* gives

$$H : H' :: d\ f : d'\ f' \quad \ldots \ldots \quad (22b)$$

Combining ($22a$) and ($22b$), we have

$$H : H' :: o\ d : o\ d' \quad \ldots \ldots \quad (22c)$$

or, the horizontal forces are directly proportional to their distances from the neutral axis.

Denote the distance of any fibre from the neutral axis by z, the stress in it by H', the distance of the remotest fibre by e, and its stress by H; then, from ($22c$), we obtain

$$H' : H :: z : e, \text{ or, } H' = \frac{H\ z}{e} \quad \ldots \ldots \quad (22d)$$

Thus far the cross-section of the fibres has been considered as unity. If the actual area is a, the force is $\frac{H a z}{e}$. Each of these forces H' tend to turn the beam around o with a lever arm $o\ d'$ or z, hence the moment of the force is $\frac{H\ a\ z}{e} \times z = \frac{H\ a\ z^2}{e}$, and the sum of all the moments is

$$\overset{r}{M_x} = \frac{H}{e} \, \Sigma \, a \, z^2 \quad \dots \dots \dots \dots \quad (22e)$$

$\overset{r}{M_x}$ meaning the bending moment at any section x_r of the beam in the span l_r.

Since $\Sigma \, a \, z^2$ is the expression for the moment of inertia of the section $a \, b$, (22e) becomes

$$\overset{r}{M_x} = \frac{H \, \overset{r}{I_x}}{e} \quad \dots \dots \dots \dots \quad (22f)$$

or the moment of the internal forces equals the stress in the remotest fibre times the moment of inertia of the section divided by the distance of the remotest fibre from the neutral axis.

The line $d \, f$ denotes the change of length in the fire $a \, d$, due to the force H, hence, if E be the co-efficient of elasticity,

$$a \, d : d \, f :: E : H \quad \dots \dots \dots \dots \quad (22g)$$

Designating the radius $c \, o$ by γ_r, we have, from similar figures, $o \, d \, f$ and $c \, a \, d$. $(m \, o = a \, d)$.

$$a \, d : d \, f :: \gamma_r : e \quad \dots \dots \dots \dots \quad (22h)$$

$$\frac{H}{e} = \frac{E}{\gamma_r}, \text{ or, } e = \frac{H \, \gamma_r}{E} \quad \dots \dots \dots \dots \quad (22i)$$

Substituting this value of e in (22f), we have

$$\overset{r}{M_x} = \frac{E \, \overset{r}{I_x}}{\gamma_r} \quad \dots \dots \dots \dots \quad (22k)$$

The radius of curvature of any plane curve, whose length is u_r and co ordinates x_r and y_r, is

$$\gamma_r = \frac{d \, u_r^3}{d \, x_r \, d^2 \, y_r} \quad \dots \dots \dots \dots \quad (22l)$$

According to the third hypothesis, $d \, u_r = d \, x_r$, and (22l) becomes

$$\gamma_r = \frac{d \, x_r^2}{d^2 \, y_r} \quad \dots \dots \dots \dots \quad (22m)$$

Substituting (22m) in (22k), it becomes

$$\frac{d^2\, y_r}{d\, x_r^2} = \frac{\overset{r}{M}_x}{E\, \overset{r}{I}_x} \quad \ldots \ldots \ldots \quad (22)$$

Which is the differential equation of the elastic line, applicable to all bodies subjected to flexure which fulfill the conditions imposed by the *third hypothesis*. The values of E and I may be different for each and every section.

If (22) be integrated, it becomes

$$\frac{d\, y_r}{d\, x_r} = \frac{1}{E} \int_0^{x_r} \frac{\overset{r}{M}_x\, d\, x_r}{\overset{r}{I}_x} + C \quad \ldots \ldots \ldots \quad (23)$$

If $x_r = 0$, then $C = \frac{d\, y_r}{d\, x_r} = t_r$, and we have

$$\frac{d\, y_r}{d\, x_r} = t_r + \frac{1}{E} \int_0^{x_r} \frac{\overset{r}{M}_x\, d\, x_r}{\overset{r}{I}_x} \quad \ldots \ldots \ldots \quad (24)$$

Integrating again, (24) becomes

$$y_r = C' + t_r \int_0^{x_r} d\, x_r + \frac{1}{E} \int_0^{x_r} \frac{\overset{r}{M}_x\, d\, x_r}{\overset{r}{I}_x} \int_0^{x_r} d\, x_r \quad \ldots \ldots \quad (25)$$

If $x_r = 0$, then $C' = h_r$, and we have

$$y_r = h_r + t_r\, x_r + \frac{1}{E} \int_0^{x_r} \frac{\overset{r}{M}_x\, d\, x_r}{\overset{r}{I}_x} \int_0^{x_r} d\, x_r \quad \ldots \ldots \quad (26)$$

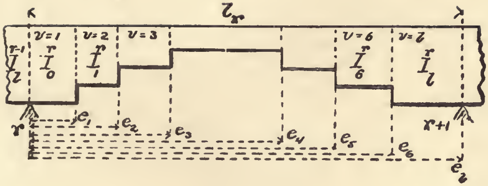


Fig. 2.

By examining Fig. 2, which represents a continuous girder having a variable cross-section, and consequently a variable moment of inertia, the following integration will be clearly understood:

* Integration of $\displaystyle\int_0^{x_r} \frac{\overset{r}{M}_x\, d\, x_r}{\overset{r}{I}_x}$

$$\int_0^{x_r} \frac{\overset{r}{M}_x\, d\, x_r}{\overset{r}{I}_x} = \frac{1}{\overset{r}{I}_0}\int_0^{e_1} \overset{r}{M}_x\, d\, x_r + \frac{1}{\overset{r}{I}_1}\int_{e_1}^{e_2} \overset{r}{M}_x\, d\, x_r \;.\;.\;.$$

$$.\;.\;.\; + \frac{1}{\overset{r}{I}_x}\int_{e_x}^{x} \overset{r}{M}_x\, d\, x_r \;.\;.\; (27)$$

But,

$$\frac{1}{\overset{r}{I}_v}\int_v^{e_{v+1}} \overset{r}{M}_x\, d\, x_r = \frac{1}{\overset{r}{I}_v}\int_0^{e_{v+1}} \overset{r}{M}_x\, d\, x_r - \frac{1}{\overset{r}{I}_v}\int_0^{e_v} \overset{r}{M}_x\, d\, x_r \;.\;.\;.\;.\;.\; (28)$$

Therefore, we can write for (27),

---

* See Weyrauch's Continuirlichen und Einfachen Träger, p. 168.

$$\int_0^{x_r} \frac{1}{\overset{r}{I}_x} \overset{r}{M}_x \, d\, x_r = \frac{1}{\overset{r}{I}_0} \int_0^{e_1} \overset{r}{M}_x \, d\, x_r + \frac{1}{\overset{r}{I}_1} \int_0^{e_2} \overset{r}{M}_x \, d\, x_r \ldots$$

$$\ldots + \frac{1}{\overset{r}{I}_{x-1}} \int_0^{e_x} \overset{r}{M}_x \, d\, x_r$$

$$- \frac{1}{\overset{r}{I}_1} \int_0^{e_1} \overset{r}{M}_x \, d\, x_r - \frac{1}{\overset{r}{I}_2} \int_0^{e_2} \overset{r}{M}_x \, d\, x_r \ldots - \frac{1}{\overset{r}{I}_x} \int_0^{e_x} \overset{r}{M}_x \, d\, x_r$$

$$+ \frac{1}{\overset{r}{I}_x} \int_0^{x_r} \overset{r}{M}_x \, d\, x_r \ldots\ldots \quad (29)$$

Which reduces to

$$\int_0^{x_r} \frac{1}{\overset{r}{I}_x} \overset{r}{M}_x \, d\, x_r = \frac{1}{\overset{r}{I}_x} \int_0^{x_r} \overset{r}{M}_x \, d\, x_r$$

$$+ \sum_{v=x_r}^{v=1} \left(\frac{1}{\overset{r}{I}_{v-1}} - \frac{1}{\overset{r}{I}_v} \right) \int_0^{e_v} \overset{r}{M}_x \, d\, x_r \ldots \quad (30)$$

Substituting (30) in (26), it reduces to

$$y_r = h_r + t_r\, x_r + \frac{1}{E} \left\{ \frac{1}{\overset{r}{I}_x} \int_0^{x_r} d\, x_r \int_0^{x_r} \overset{r}{M}_x \, d\, x_r \right.$$

$$\left. + \sum_{v=x_r}^{v=1} \left(\frac{1}{\overset{r}{I}_{v-1}} - \frac{1}{\overset{r}{I}_v} \right) \int_0^{x_r} d\, x_r \int_0^{e_v} \overset{r}{M}_x \, d\, x_r \right\} \ldots \quad (31)$$

From 8, we have $\overset{r}{M}_x = M_r + S_r\, x_r - \Sigma\, P_r\, (x_r - a_r) \;.\;.\; a \leqq x \;.\;.$ (8)

Hence,

$$\int_0^{x_r} \overset{r}{M}_x \, d\, x_r = M_r\, x_r + \frac{1}{2}\, S_r\, x_r^2 - \frac{1}{2} \Sigma P_r\, (x_r - a_r)^2 \ldots a \leqq x \ldots (32)$$

And,

$$\int_0^{x_r} d\, x_r \int_0^{x_r} \overset{r}{M}_x \, d\, x_r = \frac{1}{2} M_r\, x_r^2 + \frac{1}{6}\, S_r\, x_r^3$$

$$- \frac{1}{6}\, \Sigma\, P_r\, (x_r - a_r)^3 \;.\;.\; a \leqq x \;.\;.\; (33)$$

Now,

$$\int_0^{x_r} d\, x_r \int_0^{e_v} \overset{r}{M}_x \, d\, x_r = \int_0^{e_v} d\, x \int_0^{e_v} \overset{r}{M}_x \, d\, x_r$$

$$+ \int_{e_v}^{x_r} d\, x_r \int_0^{e_v} \overset{r}{M}_x \, d\, x_r \;.\;.\;.\;.\;.\; (34)$$

But,

$$\int_0^{e_v} \overset{r}{M}_x \, d\, x_r = M_r\, e_v + \frac{1}{2}\, S_r\, e_v^2$$

$$- \frac{1}{2}\, \Sigma\, P_r\, (e_v - a_r)^2 \;.\;.\;.\; a \leqq e \;.\;.\;.\; (35)$$

Hence,

$$\int_0^{e_v} d\, x_r \int_0^{e_v} \overset{r}{M}_x \, d\, x_r = \frac{1}{2}\, M_r\, e_v^2 + \frac{1}{6}\, S_r\, e_v^3$$

$$- \frac{1}{6}\, \Sigma\, P_r\, (e_v - a_r)^3 \;.\;.\;.\; a \leqq e \;.\;.\;.\; (36)$$

$$\int_{e_v}^{x_r} d\, x_r = x_r - e_v,$$

Therefore,

$$\int_{e_r}^{x_r} d\,x_r \int_{0}^{e_v} \overset{r}{M}_x\, d\,x_r = \frac{1}{2}\,(x_r - e_v)\left\{2\,M_r\,e_r + S_r\,e_r^2 - \Sigma\,P_r\,(e_r - a_r)^2\right\} \quad . \; . \; . \; . \; . \; (37)$$

Substituting (33), (34), (36) and (37) in (31), it becomes

$$y_r = h_r + t_r\,x_r + \frac{1}{6\,E\,\overset{r}{I}_x}\left\{3\,M_r\,x_r^2 + S_r\,x_r^3 - \Sigma\,P_r\,(x_r - a_r)^3\right\}$$

$$+ \frac{1}{6\,E\,\overset{r}{I}_x}\sum_{v=x_r}^{v=1}\left(\frac{\overset{r}{I}_x}{\overset{r}{I}_{v-1}} - \frac{\overset{r}{I}_x}{\overset{r}{I}_v}\right)\left\{3\,M_r\,e_v^2 + S_r\,e_v^3 - \Sigma\,P_r\,(e_v - a_r)^3\right.$$

$$\left. + 3\,(x_r - e_v)\left[2\,M_r\,e_v + S_r\,e_v^2 - \Sigma\,P_r\,(e_v - a_r)^2\right]\right\} \quad . \; . \; . \; (38)$$

Which reduces to

$$y_r = h_r + t_r\,x_r + \frac{1}{6\,E\,\overset{r}{I}_x}\left\{3\,M_r\,x_r^2 + S_r\,x_r^3 - \Sigma\,P_r\,(x_r - a_r)^3\right\}$$

$$+ \frac{1}{6\,E\,\overset{r}{I}_x}\sum_{v=x_r}^{v=1}\left(\frac{\overset{r}{I}_x}{\overset{r}{I}_{v-1}} - \frac{\overset{r}{I}_x}{\overset{r}{I}_v}\right)\left\{3\,M_r\,e_v\,(2\,x_r - e_r) + S_r\,e_v^2\,(3\,x_r - e_r)\right.$$

$$\left. - \Sigma\,P_r\,(e_v - a_r)^3 - 3\,(x_r - e_v)\,\Sigma\,P_r\,(e_r - a_r)^2\right\} \quad . \; . \; . \; . \; (39)$$

If we make $x_r = l_r$, then $y_r = h_{r+1}$, $e_x - e_l$, and $\overset{r}{I}_x = \overset{r}{I}_l$.

Let $\dfrac{\overset{r}{I}_l}{\overset{r}{I}_{v-1}} - \dfrac{\overset{r}{I}_l}{\overset{r}{I}_v} = \overset{r}{\triangle}_v$, then, from (39), we have

$$h_{r+1} = h_r + t_r\,l_r + \frac{1}{6\,E\,\overset{r}{I}_l}\left\{3\,M_r\,l_r^2 + S_r\,l_r^3 - \Sigma\,P_r\,(l_r - a_r)^3\right\}$$

$$+ \frac{1}{6\,E\,\overset{r}{I}_l}\sum_{v=l_r}^{v=1}\overset{r}{\triangle}_v\left\{3\,M_r\,e_v\,(2\,l_r - e_v) + S_r\,e_v^2\,(3\,l_r - 2\,e_v)\right.$$

$$\left. - \Sigma\,P_r\,(e_v - a_r)^3 - 3\,(l_r - e_v)\,\Sigma\,P_r\,(e_r - a_r)^2\right\} \quad . \; . \; . \; (40)$$

From (10),

$$S_r = \frac{M_{r+1} - M_r + \Sigma\,P_r\,l_r\,(1 - k_r)}{l_r} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (10)$$

But, since $k_r = \frac{a_r}{l_r}$, this becomes

$$S_r = \frac{M_{r+1} - M_r + \Sigma\ P_r\ (l_r - a_r)}{l_r} \qquad \ldots\ldots\ldots\ldots \quad (10a)$$

Substituting $(10a)$ in (40), and solving for t_r, we obtain

$$t_r = \frac{h_{r+1} - h_r}{l_r} - \frac{1}{6\ E\ \overset{r}{I_l}\ l_r} \Big\{ 2\ M_r\ l_r^2 + M_{r+1}\ l_r^2$$

$$+ \Sigma\ P_r\ a_r\ (l_r - a_r)\ (2\ l_r - a_r) \Big\}$$

$$- \frac{1}{6\ E\ \overset{r}{I_l}\ l_r} \sum_{v=l_r}^{v=1} \overset{r}{\triangle}_v \Big\{ 2\ M_r\ e_v\ (3\ l_r - 3\ e_v - \frac{e_v^2}{l_r})$$

$$+ M_{r+1}\ e_v^2\ (3 - \frac{2\,e_v}{l_r})$$

$$+ e_v^2\ (3 - \frac{2\,e_v}{l_r})\ \Sigma\ P_r\ (l_r - a_r) - \Sigma\ P_r\ (e_v - a_r)^3$$

$$- 3\ (l_r - e_v)\ \Sigma\ P_r\ (e_v - a_r)^2 \Big\} \qquad \ldots\ldots \quad (41)$$

Since $a_r = k_r\ l_r$, $a_r\ (l_r - a_r)\ (2\ l_r - a_r) = l_r^3\ (2\ k_r - 3\ k_r^2 + k_r^3)$, $l_r - a_r = l_r\ (1 - k_r)$, and (41) reduces to

$$t_r = \frac{h_{r+1} - h_r}{l_r} - \frac{1}{6\ E\ \overset{r}{I_l}} \Big\{ 2\ M_r\ l_r + M_{r+1}\ l_r$$

$$+ \Sigma\ P_r\ l_r^2\ (2\ k_r - 3\ k_r^2 + k_r^3) \Big\}$$

$$- \frac{1}{6\ E\ \overset{r}{I_l}\ l_r} \sum_{v=l_r}^{v=1} \overset{r}{\triangle}_v \Big\{ 2\ M_r\ e_v\ (3\ l_r - 3\ e_v + \frac{e_v^2}{l_r}) + M_{r+1}\ e_v^2\ (3 - \frac{2\,e_v}{l_r})$$

$$+ e_v^2\ (3 - \frac{2\,e_v}{l_r})\ \Sigma\ P_r\ l_r\ (1 - k_r) - \Sigma\ P_r\ (e_v - a_r)^3$$

$$- 3\ (l_r - e_v)\ \Sigma\ P_r\ (e_v - a_r)^2 \Big\} \qquad \ldots\ldots \quad (42)$$

Returning to (24),

$$\frac{d\ y_r}{d\ x_r} = t_r + \frac{1}{E} \int_0^{x_r} \frac{\overset{r}{M_x}\ d\ x_r}{\overset{r}{I_x}} \qquad \ldots\ldots\ldots\ldots \quad (24)$$

Substituting (32) and (35) in (30), and the result in (24), it becomes

$$\frac{d\,y_r}{d\,x_r} = t_r + \frac{1}{2\ E\ \overset{r}{I}_x} \left\{ 2\ M_r\ x_r + S_r\ x_r^2 - \Sigma\ P_r\ (x_r - a_r)^2 \right\}$$

$$+ \frac{1}{2\ E\ \overset{r}{I}_x} \sum_{v=x_r}^{v=1} \left(\frac{\overset{r}{I}_x}{\overset{r}{I}_{v-1}} - \frac{\overset{r}{I}_x}{\overset{r}{I}_r} \right)$$

$$\left\{ 2\ M_r\ e_v + S_r\ e_v^2 - \Sigma\ P_r\ (e_r - a_r)^2 \right\} \quad . \quad . \quad . \quad . \ (43)$$

Making $x_r = l_r$, then $\frac{d\ y_r}{d\ x_r} = t_{r+1}$, $e_x = e_l$, and $\overset{r}{I}_x = \overset{r}{I}_l$, and substituting for S_r its value from (10), t_r from (42), and $k_r\ l_r$ for a_r in (43), it becomes

$$t_{r+1} = \frac{h_{r+1} - h_r}{l_r} - \frac{1}{6\ E\ \overset{r}{I}_l} \left\{ 2\ M_r\ l_r + M_{r+1}\ l_r \right.$$

$$\left. + \ P_r\ l_r^2\ (2\ k_r - 3\ k_r^2 + k_r^3) \right\}$$

$$- \frac{1}{6\ E\ \overset{r}{I}_l\ l_r} \sum_{v=l_r}^{v=1} \overset{r}{\triangle}_r \left\{ 2\ M_r\ e_r\ (3\ l_r - 3\ e_r + \frac{e_r^2}{l_r}) \right.$$

$$+ M_{r+1}\ e_r^2\ (3 - \frac{2\ e_v}{l_r})$$

$$+ e_r^2\ (3 - \frac{2\ e_v}{l_r})\ \Sigma\ P_r\ l_r\ (1 - k_r) - \Sigma\ P_r\ (e_v - a_r)^3$$

$$\left. - 3\ (l_r - e_r)\ \Sigma\ P_r\ (e_v - a_r)^2 \right\}$$

$$+ \frac{1}{6\ E\ \overset{r}{I}_l} \left\{ 3\ M_r\ l_r + 3\ M_{r+1}\ l_r + 3\ \Sigma\ P_r\ l_r^2\ (k_r - k_r^2) \right\}$$

$$+ \frac{1}{6\ E\ \overset{r}{I}_l\ l_r} \sum_{v=l_r}^{v=1} \overset{r}{\triangle}_v \left\{ 3\ M_r\ e_v\ l_r\ (2 - \frac{e_v}{l_r}) + 3\ M_{r+1}\ e_v^2 \right.$$

$$\left. + 3\ \Sigma\ P_r\ (1 - k_r)\ e_v^2\ l_r - 3\ l_r\ \Sigma\ P_r\ (e_v - a_r)^2 \right\} \quad . \quad . \quad . \quad . \ (44)$$

Which reduces to

$$t_{r+1} = \frac{h_{r+1} - h_r}{l_r} + \frac{1}{6\ E\ \overset{r}{I}_l} \left\{ M_r\ l_r + 2\ M_{r+1}\ l_r + \Sigma\ P_r\ l_r^2\ (k_r - k_r^3) \right\}$$

$$+\frac{1}{6\,E\,\overset{r}{I_l}\,l_r}\sum_{v=l_r}^{v=1}\overset{r}{\triangle}_r\left\{M_r\,e_v^2\left(3-\frac{2\,e_v}{l_r}\right)+2\,M_{r+1}\frac{e_v^3}{l_r}\right.$$

$$\left.+\frac{2\,e_v^3}{l_r}\Sigma P_r\,l_r\,(1-k_r)-3\,e_v\,\Sigma\,P_r\,(e_v-a_r)^2+\Sigma\,P_r\,(e_v-a_r)^3\right\} \quad (45)$$

If we were to suppose loads in the $r-1^{th}$ span at distances $a_{r-1}=k_{r-1}\,l_{r-1}$ from the left support $r-1$, we would find in a similar manner, or by *decreasing* the subscripts of (45) by unity,

$$t_r=\frac{h_r-h_{r-1}}{l_{r-1}}+\frac{1}{6\,E\,\overset{r-1}{I_l}}\left\{M_{r-1}\,l_{r-1}+2\,M_r\,l_{r-1}\right.$$
$$\left.+\Sigma\,P_{r-1}\,l_{r-1}^2\,(k_{r-1}-k_{r-1}^3)\right\}$$

$$+\frac{1}{6\,E\,\overset{r-1}{I_l}\,l_{r-1}}\sum_{v=l_{r-1}}^{v=1}\overset{r-1}{\triangle}_v\left\{M_{r-1}\,e_v^2\left(3-\frac{2\,e_v}{l_{r-1}}\right)\right.$$

$$+2\,M_r\,\frac{e_v^3}{l_{r-1}}+\frac{2\,e_v^3}{l_{r-1}}\,\Sigma\,P_{r-1}\,l_{r-1}\,(1-k_{r-1})$$

$$\left.-3\,e_v\,\Sigma\,P_{r-1}\,(e_v+a_{r-1})^2+\Sigma\,P_{r-1}\,(e_v-a_{r-1})^3\right\}\quad . \; . \; . \; (46)$$

Equating (42) and (46), we obtain

$$\frac{h_{r+1}-h_r}{l_r}+\frac{1}{6\,E\,\overset{r}{I_l}}\left\{-2\,M_r\,l_r-M_{r+1}\,l_r-\Sigma\,P_r\,l_r^2\,(2\,k_r-3\,k_r^2+h_r^3)\right.$$

$$-\sum_{v=l_r}^{v=1}\overset{r}{\triangle}_v\,e_v\left(3-\frac{3\,e_v}{l_r}+\frac{e_v^2}{l_r^2}\right)2\,M_r-\sum_{v=l_r}^{v=1}\overset{r}{\triangle}_v\,e_v^2$$

$$\left(\frac{3}{l_r}-\frac{2\,e_v}{l_r^2}\right)M_{r+1}-\sum_{v=l_r}^{v=1}\overset{r}{\triangle}_v\,e_v^2\left(\frac{3}{l_r}-\frac{2\,e_v}{l_r^2}\right)$$

$$\Sigma\,P_r\,l_r\,(1-k_r)+\sum_{v=l_r}^{v=1}\overset{r}{\triangle}_r\,\frac{1}{l_r}\,\Sigma\,P_r\,(e_r-a_r)^3+\sum_{v=l_r}^{v=1}\overset{r}{\triangle}_v$$

$$\left.\left(\frac{3\,(l_r-e_v)}{l_r}\right)\Sigma\,P_r\,(e_v-a_r)^2\right\}=\frac{h_r-h_{r-1}}{l_{r-1}}+\frac{1}{6\,E\,\overset{r-1}{I_l}}$$

$$\Big\{ M_{r-1}\, l_{r-1} + 2\, M_r\, l_{r-1} + \Sigma\, P_{r-1}\, l_{r-1}^2\, (k_{r-1} - k_{r-1}^3) + \sum_{v=l_{r-1}}^{v=1} \overset{r-1}{\triangle}_v\, e_v^2$$

$$\left(\frac{3}{l_{r-1}} - \frac{2\, e_v}{l_{r-1}^2} \right) M_{r-1} + \sum_{v=l_{r-1}}^{v=1} \overset{r-1}{\triangle}_v \frac{e_v^3}{l_{r-1}^2}\, 2\, M_r + \sum_{v=l_{r-1}}^{v=1} \overset{r-1}{\triangle}_v \frac{2\, e_v^3}{l_{r-1}^2}$$

$$\Sigma\, P_{r-1}\, l_{r-1}\, (1 - k_{r-1}) - \sum_{v=l_{r-1}}^{v=1} \overset{r-1}{\triangle}_v \frac{3\, e_v}{l_{r-1}}\, \Sigma\, P_{r-1}\, (e_v - a_{r-1})^2$$

$$+ \sum_{v=l_{r-1}}^{v=1} \overset{r-1}{\triangle}_v \frac{1}{l_{r-1}}\, \Sigma\, P_{r-1}\, (e_v - a_{r-1})^3 \Big\} \quad . \quad . \quad . \quad . \quad (47)$$

Let

$$\sum_{v=l_r}^{v=1} \overset{r}{\triangle}_v \left(3 - \frac{3\, e_v}{l_r} + \frac{e_v^2}{l_r^2} \right) = F_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

$$\sum_{v=l_r}^{v=1} \overset{r}{\triangle}_v \left(\frac{3}{l_r} - \frac{2\, e_v}{l_r^2} \right) e_v^2 = F_r' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

$$\sum_{v=l_r}^{v=1} \overset{r}{\triangle}_v \frac{e_v^3}{l_r} = F_r'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

$$6\, E\, \overset{r}{I_l} = \theta_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$\sum_{v=l_r}^{v=1} \overset{r}{\triangle}_v \left\{ \frac{\Sigma\, P_r\, (e_v - a_r)^3}{l_r} + \frac{3\, (l_r - e_v)}{l_r}\, \Sigma\, P_r\, (e_v - a_r)^2 \right\} = H_r \quad . \quad . \quad (52)$$

$$\sum_{v=l_r}^{v=1} \overset{r}{\triangle}_v \left\{ \frac{\Sigma\, P_r\, (e_v - a_r)^3}{l_r} - \frac{3\, e_v}{l_r}\, \Sigma\, P_r\, (e_v - a_r)^2 \right\} = H_r' \quad . \quad . \quad (53)$$

Substituting (48), (49), (50), (51), (52) and (53) in (47), transposing and reducing, we obtain

$$-\theta_r\,\theta_{r-1}\,\frac{h_{r-1}-h_r}{l_r}+2\,M_r\,l_r\,\theta_{r-1}+M_{r+1}\,l_r\,\theta_{r-1}+\Sigma\,P_r\,l_r^2\,(2\,k_r-3\,k_r^2$$
$$+k_r^3)\,\theta_{r-1}+F_r\,2\,M_r\,\theta_{r-1}+F_r'\,M_{r+1}\,\theta_{r-1}+F_r'\,\Sigma\,P_r\,l_r\,(1-k_r)\,\theta_{r-1}$$
$$-H_r\,\theta_{r-1}+\frac{h_r+h_{r-1}}{l_{r-1}}\,\theta_r\,\theta_{r-1}+M_{r-1}\,l_{r-1}\,\theta_r+2\,M_r\,l_{r-1}\,\theta_r+$$
$$\Sigma\,P_{r-1}\,l_{r-1}^2\,(k_{r-1}-k_{r-1}^3)\,\theta_r+F_{r-1}'\,M_{r-1}\,\theta_r+F_{r-1}''\,2\,M_r\,\theta_r+F_{r-1}''$$
$$2\,\Sigma\,P_{r-1}\,l_{r-1}\,(1-k_{r-1})\,\theta_r+H_{r-1}'\,\theta_r=0 \quad \ldots\ldots\ldots \quad (54)$$

Which reduces to

$$M_{r-1}\,(l_{r-1}+F_{r-1}')\,\theta_r+2\,M_r\left\{\,(l_r+F_r)\,\theta_{r-1}+(l_{r-1}+F_{r-1}'')\,\theta_r\,\right\}$$
$$+M_{r+1}\,(l_r+F_r')\,\theta_{r-1}=-\,\theta_r\,\theta_{r-1}\left\{\frac{h_r-h_{r-1}}{l_{r-1}}+\frac{h_r-h_{r+1}}{l_r}\right\}$$
$$+\,\Sigma\,P_r\,l_r^2\,(2\,k_r-3\,k_r^2+k_r^3)\,\theta_{r-1}-F_r'\,\Sigma\,P_r\,l_r\,(1-k_r)\,\theta_{r-1}$$
$$-\,\Sigma\,P_{r-1}\,l_{r-1}^2\,(k_{r-1}-k_{r-1}^3)\,\theta_r-F_{r-1}''\,2\,\Sigma\,P_{r-1}\,l_{r-1}\,(1-k_{r-1})\,\theta_r$$
$$+\,H_r\,\theta_{r-1}-H_{r-1}'\,\theta_r \quad \ldots\ldots\ldots \quad (55)$$

Let

$$-\,\theta_r\,\theta_{r-1}\left\{\frac{h_r-h_{r-1}}{l_{r-1}}+\frac{h_r-h_{r+1}}{l_r}\right\}=Y_r \quad \ldots\ldots \quad (56)$$
$$-\,\Sigma\,P_r\,l_r^2\,(2\,k_r-3\,k_r^2+k_r^3)\,\theta_{r-1}=A_r \quad \ldots\ldots \quad (57)$$
$$-\,\Sigma\,P_r\,l_r^2\,(k_r-k_r^3)\,\theta_{r+1}=B_r \quad \ldots\ldots \quad (58)$$
$$\left.\begin{aligned}&-F_r'\,\Sigma\,P_r\,l_r\,(1-k_r)\,\theta_{r-1}+H_r\,\theta_{r-1}=X_r' \quad \ldots \quad (59)\\&-2\,F_{r-1}''\,\Sigma\,P_{r-1}\,l_{r-1}\,(1-k_{r-1})\,\theta_r-H_{r-1}'\,\theta_r=X_{r-1}'' \quad \ldots \quad (59a)\end{aligned}\right\}=X_r$$
$$\theta_{r+1}\,(l_r+F_r')=\beta_r \quad \ldots\ldots \quad (60)$$
$$\theta_r\,(l_{r-1}+F_{r-1}'')+\theta_{r-1}\,(l_r+F_r)=\beta_r' \quad \ldots\ldots \quad (61)$$
$$\theta_{r-1}\,(l_r+F_r')=\beta_r'' \quad \ldots\ldots \quad (62)$$

Then we can write

$$M_{r-1}\,\beta_{r-1}+2\,M_r\,\beta_r'+M_{r+1}\,\beta_r''=A_r+B_{r-1}+Y_r+X_r \quad \ldots \quad (63)$$

Which is the general form of the *Theorem of three moments.* It expresses the relation between the bending moments over three consecutive supports in terms of the loads, spans, E, I and h. An equation of the form of (63) can be written for every support, and, as, if the girder *rests* on the supports, the moment at the first and last support is zero, we shall

have as many equations as there are unknown quantities or bending moments, and hence we can determine their values.

Let there be s spans in the continuous girder represented in Fig. 1, and let the r^{th} span alone be loaded, then by the three moment theorem, the equation for each support is as follows:

$$\left.\begin{array}{l}
2\,M_2\ \beta_2' + M_3\ \beta_2'' = 0 \\
M_2\ \beta_2 + 2\,M_3\ \beta_3' + M_4\ \beta_3'' = 0 \\
\times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \\
M_{r-1}\,\beta_{r-1} + 2\,M_r\ \beta_r' + M_{r+1}\,\beta_r'' = Y_r + A_r + X_r \\
M_r\ \beta_r + 2\,M_{r+1}\,\beta_{r+1}' + M_{r+2}\,\beta_{r+1}'' = B_r \\
\times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \\
M_{s-2}\,\beta_{s-2} + 2\,M_{s-1}\,\beta_{s-1}' + M_s\ \beta_{s-1}'' = 0 \\
M_{s-1}\,\beta_{s-1} + 2\,M_s\ \beta_s' = 0
\end{array}\right\} \quad . \quad . \quad (64)$$

Multiplying the first equation of (64) by c_2, the second by c_3, the r^{th} by c_r, etc, we obtain, after reduction,

$$\begin{array}{l}
M_2\,(2\,c_2\,\beta_2' + c_3\,\beta_2) + M_3\,(c_2\,\beta_2'' + 2\,c_3\,\beta_3' + c_4\,\beta_3) \\
\qquad\qquad + M_4\,(c_3\,\beta_3'' + 2\,c_4\,\beta_4' + c_5\,\beta_4) \\
\times \quad \times \quad \times \quad \times \quad \times \quad \times \\
\quad + M_{s-1}\,(c_{s-2}\,\beta_{s-2}'' + 2\,c_{s-1}\,\beta_{s-1}' + c_s\,\beta_{s-1} \\
\quad + M_s\,(\beta_{s-1}''\,c_{s-1} + 2\,c_s\,\beta_s') = Y_r\,c_r + X_r\,c_r + A_r\,c_r\ B_r\,c_{r+1} \quad . \quad . \quad (65)
\end{array}$$

Now, supposing it is desired to determine the value of M_s, it is only necessary to impose such conditions upon the multiplier c that all terms shall reduce to zero, excepting those containing M_s. Evidently, the co-efficients must separately equal zero, or

$$\left.\begin{array}{l}
2\,c_2\,\beta_2' + c_3\,\beta_2 = 0 \\
c_2\,\beta_2'' + 2\,c_3\,\beta_3' + c_4\,\beta_3 = 0 \\
c_3\,\beta_3'' + 2\,c_4\,\beta_4' + c_5\,\beta_4 = 0 \\
\times \quad \times \quad \times \quad \times \\
c_{m-1}\,\beta_{m-1}'' + 2\,c_m\,\beta_m' + c_{m+1}\,\beta_m = 0
\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

And, for M_s, we have, at once,

$$M_s = \frac{(Y_r + X_r)\,c_r + A_r\,c_r + B_r\,c_{r+1}}{(c_{s-1}\,\beta_{s-1}'' + 2\,c_s\,\beta_s') = -c_{s+1}\,\beta_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

In a similar manner, multiplying the last equation of (64) by d_2, the last but one by d_3, and so on, we obtain

$$\left.\begin{array}{l} \qquad 2\, d_2\, \beta'_s \quad + \; d_3\, \beta''_{s-1} = 0 \\ d_2\, \beta_{s-1} + 2\, d_3\, \beta'_{s-1} + d_4\, \beta''_{s-2} = 0 \\ d_3\, \beta_{s-2} + 2\, d_4\, \beta'_{s-2} + d_5\, \beta''_{s-3} = 0 \\ \quad \times \qquad \times \qquad \times \qquad \times \\ d_{m-1}\, \beta_{s-m+2} + 2\, d_m\, \beta'_{s-m+2} + d_{m+1}\, \beta''_{s-m+1} = 0 \end{array}\right\} \quad \ldots\ldots \quad (68)$$

And

$$M_2 = \frac{(Y_r + X_r)\, d_{s-r+2} + A_r\, d_{s-r+2} + B_r\, d_{s-r+1}}{(2\, \beta'_2\, d_s + d_{s-1}\, \beta_2) = -\, d_{s+1}\, \beta''_1} \quad \ldots\ldots \quad (69)$$

From (65), we see that $M_3 = -\dfrac{2\, \beta'_2}{\beta''_2} M_2$, from (66), $c_3 = -\dfrac{2\, c_2\, \beta'_3}{\beta_3}$; hence, assuming $c_1 = 0$ and $c_2 = 1$, we have $M_3 = c_3 \dfrac{\beta_2}{\beta''_2} M_2$, and in a similar manner we find that $M_4 = c_4 \dfrac{\beta_2\, \beta_3}{\beta''_2\, \beta''_3} M_2$, $M_5 = c_5 \dfrac{\beta_2\, \beta_3\, \beta_4}{\beta''_2\, \beta''_3\, \beta''_4} M_2$, or in general for any support *on the left* of the loaded span or spans.

$m < r + 1$

$$M_m = c_m \frac{\beta_2\, \beta_3 \,.\,.\, \beta_{m-1}}{\beta''_2\, \beta''_3 \,.\,.\, \beta_{m-1}} M_2 \quad \ldots\ldots\ldots\ldots \quad (70)$$

In a like manner, we obtain for any support *on the right* of the loaded span or spans,

$m > r$

$$M_m = d_{s-m+2} \frac{\beta''_{s-1}\, \beta''_{s-2} \,.\,.\, \beta''_m}{\beta_{s-1}\, \beta_{s-2} \,.\,.\, \beta_m} M_s \quad \ldots\ldots\ldots\ldots \quad (71)$$

Substituting (69) in (70), and (67) in (71), we obtain

$m < r + 1$

$$M_m = -\frac{c_m\, \beta_2\, \beta_3 \,.\,.\, \beta_{m-1}}{(d_{s+1}\, \beta''_1)\, \beta''_2 \,.\,.\, \beta''_{m-1}} \left\{ (A_r + Y_r + X_r)\, d_{s-r+2}\, B_r\, d_{s-r+1} \right\} \; (72)$$

$$M_m = -\frac{d_{s-m+2}\, \beta''_{s-1}\, \beta''_{s-2} \,.\,.\, \beta''_m}{(c_{s+1}\, \beta_3)\, \beta_{s-1}\, \beta_{s-2} \,.\,.\, \beta_m} \left\{ (A_r + Y_r + X_r)\; c_r + B_r\, c_{r+1} \right\} \ldots (73)$$

In (72), as m must always be *less* than $r + 1$, r can have values from s to m.

In (73), as m must always be *greater* than r, r can have values from 1 to $m-1$.

Hence, adding (72) and (73), and substituting $X_r'+X_{r-1}''$ for X_r, we have

$$M_m = \frac{-c_m \beta_2 \beta_3 \ldots \beta_{m-1}}{(d_{s+1} \beta_1'') \beta_2'' \beta_3'' \ldots \beta_{m-1}''} \sum_{r=s}^{r=m} \left\{ (A_r + Y_r + X_r' + X_{r-1}'') d_{s-r+2} + B_r d_{s-r+1} \right\}$$

$$+ \frac{-d_{s-m+2} \beta_{s-1}'' \beta_{s-2}'' \ldots \beta_m''}{(c_{s+1} \beta_s) \beta_{s-1} \beta_{s-2} \ldots \beta_m} \sum_{r=m-1}^{r=1} \left\{ (A_r + Y_r + X_r' + X_{r-1}'') c_r + B_1 c_{r+1} \right\} \quad (A)$$

From (A), we can obtain the bending moment over any support m of a continuous girder of any number of spans s, of any lengths as $l_1, l_2 \;.\;.\;.\; l_s$, supports at any levels, the moment of inertia I constant or variable, the modulus of elasticity E being constant, and the loads being placed at pleasure.

Note that $\beta_2'' \leqq \beta_{m-1}''$, $\beta_2 \leqq \beta_{m-1}$, $\beta_{s-1}'' \geqq \beta_m''$, and $\beta_{s-1} \geqq \beta_m$.

$k=\frac{a}{l}$	$k-k^3$		$k=\frac{a}{l}$	$k-k^3$		$k=\frac{a}{l}$	$k-k^3$	
			.060	059 784 000	.940	.120	118 272 000	.880
.001	000 999 999	.999	61	060 773 019	39	21	119 228 439	79
2	001 999 992	8	62	061 761 672	38	22	120 184 152	78
3	002 999 973	7	63	062 749 953	37	23	121 139 133	77
4	003 999 936	6	64	063 737 856	36	24	122 093 376	76
5	004 999 875	5	65	064 725 375	35	25	123 046 875	75
6	005 999 784	4	66	065 712 504	34	26	123 999 624	74
7	006 999 657	3	67	066 699 237	33	27	124 951 617	73
8	007 999 488	2	68	067 685 568	32	28	125 902 848	72
9	008 999 271	1	69	068 671 491	31	29	126 853 311	71
.010	009 989 000	.990	.070	069 657 000	.930	.130	127 803 000	.870
11	010 998 669	89	71	070 642 089	29	31	128 751 909	69
12	011 998 272	88	72	071 626 752	28	32	129 700 032	68
13	012 997 803	87	73	072 610 983	27	33	130 647 363	67
14	013 997 256	86	74	073 594 776	26	34	131 593 896	66
15	014 996 625	85	75	074 578 125	25	35	132 539 625	65
16	015 995 904	84	76	075 561 024	24	36	133 484 544	64
17	016 995 087	83	77	076 543 467	23	37	134 428 647	63
18	017 994 168	82	78	077 525 448	22	38	135 371 928	62
19	018 993 141	81	79	078 506 961	21	39	136 314 381	61
.020	019 992 000	.980	.080	079 488 000	.920	.140	137 256 000	.860
21	020 990 739	79	81	080 468 559	19	41	138 196 779	59
22	021 989 352	78	82	081 448 632	18	42	139 136 712	58
23	022 987 833	77	83	082 428 213	17	43	140 075 793	57
24	023 986 176	76	84	083 407 296	16	44	141 014 016	56
25	024 984 375	75	85	084 385 875	15	45	141 951 375	55
26	025 982 424	74	86	085 363 944	14	46	142 887 864	54
27	026 980 317	73	87	086 341 497	13	47	143 823 477	53
28	027 978 048	72	88	087 318 528	12	48	144 758 208	52
29	028 975 611	71	89	088 295 031	11	49	145 692 051	51
.030	029 973 000	.970	.090	089 271 000	.910	.150	146 625 000	.850
31	030 970 209	69	91	090 246 429	9	51	147 557 049	49
32	031 967 232	68	92	091 221 312	8	52	148 488 192	48
33	032 964 063	67	93	092 195 643	7	53	149 418 423	47
34	033 960 696	66	94	093 169 416	6	54	150 347 736	46
35	034 957 125	65	95	094 142 625	5	55	151 276 125	45
36	035 953 344	64	96	095 115 264	4	56	152 203 584	44
37	036 949 347	63	97	096 087 327	3	57	153 130 107	43
38	037 945 128	62	98	097 058 808	2	58	154 055 688	42
39	038 940 681	61	99	098 029 701	1	59	154 980 321	41
.040	039 936 000	.960	.100	099 000 000	.900	.160	155 904 000	.840
41	040 931 079	59	1	099 969 699	899	61	156 826 719	39
42	041 925 912	58	2	100 938 792	98	62	157 748 472	38
43	042 920 493	57	3	101 907 273	97	63	158 669 253	37
44	043 914 816	56	4	102 875 136	96	64	159 589 056	36
45	044 908 875	55	5	103 842 375	95	65	160 507 875	35
46	045 902 664	54	6	104 808 984	94	66	161 425 704	34
47	046 896 177	53	7	105 774 957	93	67	162 342 537	33
48	047 889 408	52	8	106 740 288	92	68	163 258 368	32
49	048 882 351	51	9	107 704 971	91	69	164 173 191	31
.050	049 875 000	.950	.110	108 669 000	.890	.170	165 087 000	.830
51	050 867 349	49	11	109 632 369	89	71	165 999 789	29
52	051 859 392	48	12	110 595 072	88	72	166 911 552	28
53	052 851 123	47	13	111 557 103	87	73	167 822 283	27
54	053 842 536	46	14	112 518 456	86	74	168 731 976	26
55	054 833 625	45	15	113 479 125	85	75	169 640 625	25
56	055 824 384	44	16	114 439 104	84	76	170 548 224	24
57	056 814 807	43	17	115 398 387	83	77	171 454 767	23
58	057 804 888	42	18	116 356 968	82	78	172 360 248	22
59	058 794 621	41	19	117 314 841	81	79	173 264 661	21
	$2k-3k^2+k^3$	$k=\frac{a}{l}$		$2k-3k^2+k^3$	$k=\frac{a}{l}$		$2k-3k^2+k^3$	$k=\frac{a}{l}$

$k=\frac{a}{l}$	$k-k^3$		$k=\frac{a}{l}$	$k-k^3$		$k=\frac{a}{l}$	$k-k^3$	
.180	174 168 000	.820	.240	226 176 000	.760	.300	273 000 000	.700
81	175 070 259	19	41	227 002 479	59	1	273 729 099	.699
82	175 971 432	18	42	227 827 512	58	2	274 456 392	98
83	176 871 513	17	43	228 651 093	57	3	275 181 873	97
84	177 770 496	16	44	229 473 216	56	4	275 905 536	96
85	178 668 375	15	45	230 293 875	55	5	276 627 375	95
86	179 565 144	14	46	231 113 064	54	6	277 347 384	94
87	180 460 797	13	47	231 930 777	53	7	278 065 557	93
88	181 355 328	12	48	232 747 008	52	8	278 781 888	92
89	182 248 731	11	49	233 561 751	51	9	279 496 371	91
.190	183 141 000	.810	.250	234 375 000	.750	.310	280 209 000	.690
91	184 032 129	9	51	235 186 749	49	11	280 919 769	89
92	184 922 112	8	52	235 996 992	48	12	281 628 672	88
93	185 810 943	7	53	236 805 723	47	13	282 335 703	87
94	186 698 616	6	54	237 612 936	46	14	283 040 856	86
95	187 585 125	5	55	238 418 625	45	15	283 744 125	85
96	188 470 464	4	56	239 222 784	44	16	284 445 504	84
97	189 354 627	3	57	240 025 407	43	17	285 144 987	83
98	190 237 608	2	58	240 826 488	42	18	285 842 568	82
99	191 119 401	1	59	241 626 021	41	19	286 538 241	81
.200	192 000 000	.800	.260	242 424 000	.740	.320	287 232 000	.680
1	192 879 399	.799	61	243 220 419	39	21	287 923 839	79
2	193 757 592	98	62	244 015 272	38	22	288 613 752	78
3	194 634 573	97	63	244 808 553	37	23	289 301 733	77
4	195 510 336	96	64	245 600 256	36	24	289 987 776	76
5	196 384 875	95	65	246 390 375	35	25	290 671 875	75
6	197 258 184	94	66	247 178 904	34	26	291 354 024	74
7	198 130 257	93	67	247 965 837	33	27	292 034 217	73
8	199 001 088	92	68	248 751 168	32	28	292 712 448	72
9	199 870 671	91	69	249 534 891	31	29	293 388 711	71
.210	200 739 000	.790	.270	250 317 000	.730	.330	294 063 000	.670
11	201 606 069	89	71	251 097 489	29	31	294 735 309	69
12	202 471 872	88	72	251 876 352	28	32	295 405 632	68
13	203 336 403	87	73	252 653 583	27	33	296 073 963	67
14	204 199 656	86	74	253 429 176	26	34	296 740 296	66
15	205 061 625	85	75	254 203 125	25	35	297 404 625	65
16	205 922 304	84	76	254 975 424	24	36	298 066 944	64
17	206 781 687	83	77	255 746 067	23	37	298 727 247	63
18	207 639 768	82	78	256 515 048	22	38	299 385 528	62
19	208 496 541	81	79	257 282 361	21	39	300 041 781	61
.220	209 352 000	.780	.280	258 048 000	.720	.340	300 696 000	.660
21	210 206 139	79	81	258 811 959	19	41	301 348 179	59
22	211 058 952	78	82	259 574 232	18	42	301 998 312	58
23	211 910 433	77	83	260 334 813	17	43	302 646 393	57
24	212 760 576	76	84	261 093 696	16	44	303 292 416	56
25	213 609 375	75	85	261 850 875	15	45	303 936 375	55
26	214 456 824	74	86	262 606 344	14	46	304 578 264	54
27	215 302 917	73	87	263 360 097	13	47	305 218 077	53
28	216 147 648	72	88	264 112 128	12	48	305 855 808	52
29	216 991 011	71	89	264 862 431	11	49	306 491 451	51
.230	217 833 000	.770	.290	265 611 000	.710	.350	307 125 000	.650
31	218 673 609	69	91	266 357 829	9	51	307 756 449	49
32	219 512 832	68	92	267 102 912	8	52	308 385 792	48
33	220 350 663	67	93	267 846 243	7	53	309 013 023	47
34	221 187 096	66	94	268 587 816	6	54	309 638 136	46
35	222 022 125	65	95	269 327 625	5	55	310 261 125	45
36	222 855 744	64	96	270 065 664	4	56	310 881 934	44
37	223 687 947	63	97	270 801 927	3	57	311 500 707	43
38	224 518 728	62	98	271 536 408	2	58	312 117 288	42
39	225 348 081	61	99	272 269 101	1	59	312 731 721	41
	$2k-3k^2+k^3$	$k=\frac{a}{l}$		$2k-3k^2+k^3$	$k=\frac{a}{l}$		$2k-3k^2+k^3$	$k=\frac{a}{l}$

$k=\frac{a}{l}$	$k-k^3$		$k=\frac{a}{l}$	$k-k^3$		$k=\frac{a}{l}$	$k-k^3$	
.360	313 344 000	.640	.420	345 912 000	.580	.480	408 000	.520
61	313 954 119	39	21	346 381 539	79	81	715 359	19
62	314 562 072	38	22	346 848 552	78	82	370 019 832	18
63	315 167 853	37	23	347 313 033	77	83	321 413	17
64	315 771 456	36	24	347 774 976	76	84	620 096	16
65	316 372 875	35	25	348 234 375	75	85	915 875	15
66	316 972 104	34	26	348 691 224	74	86	371 208 744	14
67	317 569 137	33	27	349 145 517	73	87	498 697	13
68	318 163 968	32	28	349 597 248	72	88	785 728	12
69	318 756 591	31	29	350 046 411	71	89	372 069 831	11
.370	319 347 000	.630	.430	350 493 000	.570	.490	351 000	.510
71	319 935 189	29	31	350 937 009	69	91	629 229	9
72	320 521 152	28	32	351 378 432	68	92	904 512	8
73	321 104 883	27	33	351 817 263	67	93	373 176 843	7
74	321 686 376	26	34	352 253 496	66	94	446 216	6
75	322 265 625	25	35	352 687 125	65	95	712 625	5
76	322 842 624	24	36	353 118 144	64	96	976 064	4
77	323 417 367	23	37	353 546 547	63	97	374 236 527	3
78	323 989 848	22	38	353 972 328	62	98	494 008	2
79	324 560 061	21	39	354 395 481	61	99	748 501	1
.380	325 128 000	.620	.440	354 816 000	.560	.500	375 000 000	.500
81	325 693 659	19	41	355 233 879	59	1	375 248 499	.499
82	326 257 032	18	42	355 649 112	58	2	493 992	98
83	326 818 113	17	43	356 061 693	57	3	736 473	97
84	327 376 896	16	44	356 471 616	56	4	975 936	96
85	327 933 375	15	45	356 878 875	55	5	376 212 375	95
86	328 487 544	14	46	357 283 464	54	6	445 784	94
87	329 039 397	13	47	357 685 377	53	7	676 157	93
88	329 588 928	12	48	358 084 608	52	8	903 488	92
89	330 136 131	11	49	358 481 151	51	9	377 127 771	91
.390	330 681 000	.610	.450	358 875 000	.550	.510	349 000	.490
91	331 223 529	9	51	359 266 149	49	11	567 169	89
92	331 763 712	8	52	359 654 592	48	12	782 272	88
93	332 301 543	7	53	360 040 323	47	13	994 303	87
94	332 837 016	6	54	360 423 336	46	14	378 203 256	86
95	333 370 125	5	55	360 803 625	45	15	409 125	85
96	333 900 864	4	56	361 181 184	44	16	611 904	84
97	334 429 227	3	57	556 007	43	17	811 587	83
98	334 955 208	2	58	928 088	42	18	379 008 168	82
99	335 478 801	1	59	362 297 421	41	19	201 641	81
.400	336 000 000	.600	.460	664 000	.540	.520	392 000	.480
1	336 518 799	.599	61	363 027 819	39	21	579 239	79
2	337 035 192	98	62	363 388 872	38	22	763 352	78
3	337 549 173	97	63	747 153	37	23	379 944 333	77
4	338 060 736	96	64	364 102 656	36	24	380 122 176	76
5	338 569 875	95	65	455 375	35	25	296 875	75
6	339 076 584	94	66	805 304	34	26	468 421	74
7	339 580 857	93	67	365 152 437	33	27	636 817	73
8	340 082 688	92	68	496 768	32	28	802 048	72
9	340 582 071	91	69	838 291	31	29	964 111	71
.410	341 079 000	.590	.470	366 177 000	.530	.530	381 123 000	.470
11	341 573 469	89	71	512 889	29	31	278 709	69
12	342 065 472	88	72	845 952	28	32	431 232	68
13	342 555 003	87	73	367 176 183	27	33	580 563	67
14	343 042 056	86	74	503 576	26	34	726 696	66
15	343 526 625	85	75	828 125	25	35	869 625	65
16	344 008 704	84	76	368 149 824	24	36	382 009 344	64
17	344 488 287	83	77	468 667	23	37	145 847	63
18	344 965 368	82	78	784 648	22	38	279 128	62
19	345 439 941	81	79	369 097 761	21	39	409 181	61
	$2k-3k^2+k^3$	$k=\frac{a}{l}$		$2k-3k^2+k^3$	$k=\frac{a}{l}$		$2k-3k^2+k^3$	$k=\frac{a}{l}$

$k=\frac{a}{l}$	$k-k^3$		$k=\frac{a}{l}$	$k-k^3$		$k=\frac{a}{l}$	$k-k^3$	
.540	536 000	.460	.600	384 000 000	.400	.660	504 000	.340
41	659 579	59	1	383 918 199	.399	61	195 219	39
42	779 912	58	2	832 792	98	62	371 882 472	38
43	896 993	57	3	743 773	97	63	565 753	37
44	383 010 816	56	4	651 136	96	64	245 056	36
45	121 375	55	5	554 875	95	65	370 920 375	35
46	228 664	54	6	454 984	94	66	591 704	34
47	332 677	53	7	351 457	93	67	259 037	33
48	433 408	52	8	244 288	92	68	369 922 368	32
49	530 851	51	9	133 471	91	69	581 691	31
.550	383 625 000	.450	.610	019 000	.390	.670	237 000	.330
51	715 849	49	11	382 900 869	89	71	368 888 289	29
52	803 392	48	12	779 072	88	72	535 552	28
53	887 623	47	13	653 603	87	73	178 783	27
54	968 536	46	14	524 456	86	74	367 817 976	26
55	384 046 125	45	15	391 625	85	75	453 125	25
56	120 384	44	16	255 104	84	76	084 224	24
57	191 307	43	17	114 887	83	77	366 711 267	23
58	250 888	42	18	381 970 968	82	78	334 248	22
59	323 121	41	19	823 341	81	79	365 953 161	21
.560	384 000	.440	.620	672 000	.380	.680	568 000	.320
61	441 519	39	21	516 939	79	81	178 759	19
62	495 672	38	22	358 152	78	82	364 785 432	18
63	546 453	37	23	195 633	77	83	388 013	17
64	593 856	36	24	029 376	76	84	363 986 496	16
65	637 875	35	25	380 859 375	75	85	580 875	15
66	678 504	34	26	685 624	74	86	171 144	14
67	715 737	33	27	508 117	73	87	362 757 297	13
68	749 568	32	28	326 848	72	88	339 328	12
69	779 991	31	29	141 811	71	89	361 917 231	11
.570	807 000	.430	.630	379 953 000	.370	.690	491 000	.310
71	830 589	29	31	760 409	69	91	060 629	9
72	850 752	28	32	564 032	68	92	360 626 112	8
73	867 483	27	33	363 863	67	93	187 443	7
74	880 776	26	34	159 896	66	94	359 744 616	6
75	890 625	25	35	378 952 125	65	95	297 625	5
76	897 024	24	36	740 544	64	96	358 846 464	4
77	899 967	23	37	525 147	63	97	391 127	3
78	899 448	22	38	305 928	62	98	357 931 608	2
79	895 461	21	39	082 881	61	99	467 901	1
.580	888 000	.420	.640	377 856 000	.360	.700	357 000 000	.300
81	877 059	19	41	625 279	59	1	356 527 899	.299
82	862 632	18	42	390 712	58	2	051 592	98
83	844 713	17	43	152 293	57	3	355 571 073	97
84	823 296	16	44	376 910 016	56	4	086 336	96
85	798 375	15	45	663 875	55	5	354 597 375	95
86	769 944	14	46	413 864	54	6	354 104 184	94
87	737 997	13	47	159 977	53	7	353 606 757	93
88	702 528	12	48	375 902 208	52	8	105 088	92
89	663 531	11	49	640 551	51	9	352 599 171	91
.590	621 000	.410	.650	375 375 000	.350	.710	089 000	.290
91	574 929	9	51	105 549	49	11	351 574 569	89
92	525 312	8	52	374 832 192	48	12	055 872	88
93	472 143	7	53	554 923	47	13	350 532 903	87
94	415 416	6	54	273 736	46	14	005 656	86
95	355 125	5	55	373 988 625	45	15	349 474 125	85
96	291 264	4	56	699 584	44	16	348 938 304	84
97	223 827	3	57	406 607	43	17	398 187	83
98	152 808	2	58	109 688	42	18	347 853 768	82
99	078 201	1	59	372 808 821	41	19	305 041	81
	$2k-3k^2+k^3$	$k=\frac{a}{l}$		$2k-3k^2+k^3$	$k=\frac{a}{l}$		$2k-3k^2+k^3$	$k=\frac{a}{l}$

$k=\frac{a}{l}$	$k-k^3$		$k=\frac{a}{l}$	$k-k^3$		$k=\frac{a}{l}$	$k-k^3$	
.720	346 752 000	.280	.780	305 448 000	.220	.840	247 296 000	.160
21	194 639	79	81	304 620 459	19	41	246 176 679	59
22	345 632 952	78	82	303 788 232	18	42	245 052 312	58
23	066 933	77	83	302 951 313	17	43	243 922 893	57
24	344 496 576	76	84	302 109 696	16	44	242 788 416	56
25	343 921 875	75	85	301 263 375	15	45	241 648 875	55
26	343 342 824	74	86	300 412 344	14	46	240 504 264	54
27	342 759 417	73	87	299 556 597	13	47	239 354 577	53
28	171 648	72	88	298 696 128	12	48	238 199 808	52
29	341 579 511	71	89	297 830 931	11	49	237 039 951	51
.730	340 983 000	.270	.790	296 961 000	.210	.850	235 875 000	.150
31	382 109	69	91	296 086 329	9	51	234 704 919	49
32	339 776 832	68	92	295 206 912	8	52	233 529 792	48
33	167 163	67	93	294 322 743	7	53	232 349 523	47
34	338 553 096	66	94	293 433 816	6	54	231 164 136	46
35	337 934 625	65	95	292 540 125	5	55	229 973 625	45
36	311 744	64	96	291 641 664	4	56	228 777 984	44
37	336 684 447	63	97	290 738 427	3	57	227 577 207	43
38	052 728	62	98	289 830 408	2	58	226 371 288	42
39	335 416 581	61	99	288 917 601	1	59	225 160 221	41
.740	334 776 000	.260	.800	288 000 000	.200	.860	223 944 000	.140
41	130 979	59	1	287 077 599	.199	61	222 722 619	39
42	333 481 512	58	2	286 150 392	98	62	221 496 072	38
43	332 827 593	57	3	285 218 373	97	63	220 264 353	37
44	169 216	56	4	284 281 536	96	64	219 027 456	36
45	331 506 375	55	5	283 339 875	95	65	217 785 375	35
46	330 839 064	54	6	282 393 384	94	66	216 538 104	34
47	167 277	53	7	281 442 057	93	67	215 285 637	33
48	329 491 008	52	8	280 485 888	92	68	214 027 968	32
49	328 810 251	51	9	279 524 871	91	69	212 765 091	31
.750	328 125 000	.250	.810	278 559 000	.190	.870	211 497 000	.130
51	327 435 249	49	11	277 588 269	89	71	210 223 689	29
52	326 740 992	48	12	276 612 672	88	72	208 945 152	28
53	326 042 223	47	13	275 632 203	87	73	207 661 383	27
54	325 338 936	46	14	274 646 856	86	74	206 372 376	26
55	324 631 125	45	15	273 656 625	85	75	205 078 125	25
56	323 918 784	44	16	272 661 504	84	76	203 778 624	24
57	323 201 907	43	17	271 661 487	83	77	202 473 867	23
58	322 480 488	42	18	270 656 568	82	78	201 163 848	22
59	321 754 521	41	19	269 646 741	81	79	199 848 561	21
.760	321 024 000	.240	.820	268 632 000	.180	.880	198 528 000	.120
61	320 288 919	39	21	267 612 339	79	81	197 202 159	19
62	319 549 272	38	22	266 587 752	78	82	195 871 032	18
63	318 805 053	37	23	265 558 233	77	83	194 534 613	17
64	318 056 256	36	24	264 523 776	76	84	193 192 896	16
65	317 302 875	35	25	263 484 375	75	85	191 845 875	15
66	316 544 904	34	26	262 440 024	74	86	190 493 544	14
67	315 782 337	33	27	261 390 717	73	87	189 135 897	13
68	315 015 168	32	28	260 336 448	72	88	187 772 928	12
69	314 243 391	31	29	259 277 211	71	89	186 404 631	11
.770	313 467 000	.230	.830	258 213 000	.170	.890	185 031 000	.110
71	312 685 989	29	31	257 143 809	69	91	183 652 029	9
72	311 900 352	28	32	256 069 632	68	92	182 267 712	8
73	311 110 083	27	33	254 990 463	67	93	180 878 043	7
74	310 315 176	26	34	253 906 296	66	94	179 483 016	6
75	309 515 625	25	35	252 817 125	65	95	178 082 625	5
76	308 711 424	24	36	251 722 944	64	96	176 676 864	4
77	307 902 567	23	37	250 623 747	63	97	175 265 727	3
78	307 089 048	22	38	249 519 528	62	98	173 849 208	2
79	306 270 861	21	39	248 410 281	61	99	172 427 301	1
	$2k-3k^2+k^3$	$k=\frac{a}{l}$		$2k-3k^2+k^3$	$k=\frac{a}{l}$		$2k-3k^2+k^3$	$k=\frac{a}{l}$

$k=\frac{a}{l}$	$k-k^3$		$k=\frac{a}{l}$	$k-k^3$		$k=\frac{a}{l}$	$k-k^3$	
.900	171 000 000	.100	.940	109 416 000	60	.980	038 808 000	20
1	169 567 299	99	41	107 762 379	59	81	036 923 859	19
2	168 129 192	98	42	106 103 112	58	82	035 033 832	18
3	166 685 673	97	43	104 438 193	57	83	033 137 913	17
4	165 236 736	96	44	102 767 616	56	84	031 236 096	16
5	163 782 375	95	45	101 091 375	55	85	029 328 375	15
6	162 322 584	94	46	099 409 464	54	86	027 414 744	14
7	160 857 357	93	47	097 721 877	53	87	025 495 197	13
8	159 386 688	92	48	096 028 608	52	88	023 569 728	12
9	157 910 571	91	49	094 329 651	51	89	021 638 331	11
.910	156 429 000	90	.950	092 625 000	50	.990	019 701 000	10
11	154 941 969	89	51	090 914 649	49	91	017 757 729	9
12	153 449 472	88	52	089 198 592	48	92	015 808 512	8
13	151 951 503	87	53	087 476 823	47	93	013 853 343	7
14	150 448 056	86	54	085 749 336	46	94	011 892 216	6
15	148 939 125	85	55	084 016 125	45	95	009 925 125	5
16	147 424 704	84	56	082 277 184	44	96	007 952 064	4
17	145 904 787	83	57	080 532 507	43	97	005 973 027	3
18	144 379 368	82	58	078 782 088	42	98	003 998 008	2
19	142 848 441	81	59	077 025 921	41	99	001 997 001	.001
.920	141 312 000	80	.960	075 264 000	40		$2k-3k^2+k^3$	$k=\frac{a}{l}$
21	139 770 039	79	61	073 496 819	39			
22	138 222 552	78	62	071 722 872	38			
23	136 669 533	77	63	069 943 653	37			
24	135 110 976	76	64	068 158 656	36			
25	133 546 875	75	65	066 367 875	35			
26	131 977 224	74	66	064 571 304	34			
27	130 402 017	73	67	062 768 937	33			
28	128 821 248	72	68	060 960 768	32			
29	127 234 911	71	69	059 146 791	31			
.930	125 643 000	70	.970	057 327 000	30			
31	124 045 509	69	71	055 501 389	29			
32	122 442 432	68	72	053 669 952	28			
33	120 833 763	67	73	051 832 683	27			
34	119 219 496	66	74	049 989 576	26			
35	117 599 625	65	75	048 140 625	25			
36	115 974 144	64	76	046 285 824	24			
37	114 343 047	63	77	044 425 167	23			
38	112 706 328	62	78	042 558 648	22			
39	111 063 981	61	79	040 686 261	21			
	$2k-3k^2+k^3$	$k=\frac{a}{l}$		$2k-3k^2+k^3$	$k=\frac{a}{l}$			

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